Tutorial #11

MAT 188 – Linear Algebra I – Fall 2015

Solutions

Problems (Please note these are from Holt's Linear Algebra Text)

6.4 - # 7 Find the matrix A that has the eigenvalues $\lambda_1 = -1, \lambda_2 = 0$, and $\lambda_3 = 1$, and the corresponding eigenvectors $(1,1,0)^T, (1,2,1)^T$ and $(-1,1,1)^T$

Solution Via the diagonalization theorem, we know that A must satisfy

$$A = \Lambda D \Lambda^{-1} = \Lambda \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Lambda^{-1} \quad \text{where} \quad \Lambda = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

It's easy to compute the inverse of Λ, we obtain

$$\Lambda^{-1} = \begin{pmatrix} -1 & 2 & -3 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix}$$

Thus

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 & -3 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 4 \\ 0 & -1 & 2 \\ -1 & 1 & -1 \end{pmatrix}$$

$\square$

Question 2 For the following matrices, compute the characteristic polynomial of A, the eigenvalues of A, the eigenvectors for each eigenvalue, the algebraic and geometric multiplicity of each eigenvalue, and determine if the matrix is diagonalizable.

$$a)A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix} \quad b)A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{pmatrix}, \quad c)A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$$

Solution To find the characteristic polynomial for each matrix, compute $P(\lambda) = \det(A - 1\lambda)$. We obtain

$$a)P(\lambda) = (\lambda-1)^2(\lambda-2) \quad b)P(\lambda) = (\lambda-2)(\lambda-1)(\lambda+1) \quad c)P(\lambda) = \lambda^2(\lambda+2)$$

We know the eigenvalues are given by the roots of the above equations, we the eigenvalues are a) $\lambda = 2, 1$, b) $\lambda = \pm 1, 2$, and c) $\lambda = 0, -2$. To find the eigenvectors, we check the kernel of the corresponding matrices, i.e. find a solution to $(A - 1\lambda)x = 0$. We find

$$a)\vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \vec{\lambda}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad b)\vec{\lambda}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \vec{\lambda}_{-1} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \vec{\lambda}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad c)\vec{\lambda}_{-2} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}, \vec{\lambda}_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \vec{\lambda}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

For a), we see the $\lambda = 2$ has geometric and algebraic multiplicity 1 and that $\lambda = 1$ has algebraic multiplicity 2, but geometric multiplicity 1. Thus the matrix is not diagonalizable. For b), we see the $\lambda = 2$ has geometric and algebraic multiplicity, $\lambda = 1$ has algebraic multiplicity 1 and geometric multiplicity 1 and $\lambda = -11$ has algebraic multiplicity 1 and geometric multiplicity 1. Thus the matrix is diagonalizable. For c), we see the $\lambda = -2$ has geometric and algebraic multiplicity 1 and that $\lambda = 0$ has algebraic multiplicity 2 and geometric multiplicity 2. Thus the matrix is diagonalizable. $\square$