MAT237 Big List

Tyler Holden
James Mracek
Department of Mathematics
University of Toronto

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Contents

There is a standing assumption that you know at least three examples for every definition we present in the course. It is also recommended that you know counterexamples when they are relevant. For example, what is a continuous function which is not uniformly continuous? Examples and counterexamples will help you better understand the concepts in this course by giving you a sandbox where you can play with theorems.

Questions itemized with a $*$ are considered easy. Questions itemized with a $**$ are considered to have medium difficulty. Questions itemized with *** are considered hard. You should expect fewer of the hard questions to show up on tests, so budget your time on these problems accordingly; solve all the easy ones first then spend the time you have leftover on the harder questions. We have also added a category of questions labelled with (†). These questions are for your enjoyment and mental stimulation only, and should not be considered examinable. Only attempt these if you have finished everything else.

1 Basic Set Theory

Theorems: DeMorgan's laws
Definitions: Sets, subsets, set builder notation, union, intersection, complement, product, indexed families of sets, functions, injective, surjective, bijective, image, preimage

*1. Let $A \subseteq S$ and $B \subseteq S$. Prove each of the following statements

   (a) $A \subseteq B$ if and only if $A \cup B = B$
   (b) $A^c \subseteq B$ if and only if $A \cup B = S$
   (c) $A \subseteq B$ if and only if $B^c \subseteq A^c$
   (d) $A \subseteq B^c$ if and only if $A \cap B = \emptyset$

*2. Let A, B, and C be subsets of S defied by statements $P(x)$, $Q(x)$, and $R(x)$, respectively. Show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

*3. Let $A_1, A_2, \ldots, A_n$ be sets. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subset A_n$ and $A_n \subseteq A_1$, then $A_1 = A_2 = A_3 = \cdots = A_n$.

**4. Let I be an index for a collection of subsets $A_i \subseteq S$, $i \in I$. Show that for every $k \in I$, $\bigcap_{i\in I} A_i \subseteq A_k$

**5. Let $f : A \to B$ be a function.

   (a) For every $X \subseteq A$, $X \subseteq f^{-1}(f(X))$
   (b) For every $Y \subseteq B$, $Y \supseteq f(f^{-1}(Y))$
   (c) If $f : A \to B$ is injective, then for every $X \subseteq A$ we have $X = f^{-1}(f(X))$
   (d) If $f : A \to B$ is surjective, then for every $Y \subseteq B$ we have $Y = f(f^{-1}(Y))$

***6. Let $f : A \to B$ be a map of sets, and let $\{X_i\}_{i\in I}$ be an indexed collection of subsets of A.

   (a) Prove that $f\left(\bigcup_{i\in I} X_i\right) = \bigcup_{i\in I} f(X_i)$
   (b) Prove that $f\left(\bigcap_{i\in I} X_i\right) \subset \bigcap_{i\in I} f(X_i)$
   (c) When does equality of sets hold in the above part?

2 Topology of $\mathbb{R}^n$

Definitions: Open, closed, not open, not closed, closure, interior, boundary, bounded

*1. Let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^3$. Compute $\mathbf{u} \times \mathbf{v}$, $\mathbf{u} \cdot \mathbf{v}$ for the following choices of $\mathbf{u}$ and $\mathbf{v}$, and state whether or not $\mathbf{u}$ and $\mathbf{v}$ are either orthogonal or colinear.

(a) $\mathbf{u} = (6, 0, -2)$, $\mathbf{v} = (0, 8, 0)$

(b) $\mathbf{u} = (1, 1, -1)$, $\mathbf{v} = (2, 4, 6)$

(c)

*2. Let $\vec{a}, \vec{b}, \vec{c}, \vec{d} \in \mathbb{R}^3$ be vectors. Determine which of the following expressions are meaningless:

(a) $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$

(b) $|\vec{a}|(\vec{b} \cdot \vec{c})$

(c) $\vec{a} \cdot (\vec{b} + \vec{c})$

(d) $(\vec{a} \cdot \vec{b})\vec{c}$

(e) $\vec{a} \cdot \vec{b} + c$

(f) $\vec{a} \cdot (\vec{b} \times \vec{c})$

(g) $(\vec{a} \cdot \vec{b}) \cdot (\vec{c} \cdot \vec{d})$

(h) $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

*3. Prove that for any $x, y \in \mathbb{R}^n$, $|x - y| \geq \big||x| - |y|\big|$. This is commonly called, "the reverse triangle inequality". It is extremely useful when you want to prove inequalities like $|(x - a)^{-1}| \leq M$ for x in some fixed closed set.

*4. Can a set be both open and closed? Prove or disprove. Can a set be neither closed nor open? Prove or disprove. The point of this question is to get you to understand that "not closed $\neq$ open" (and conversely, "not open $\neq$ closed"). See also this humorous instructional video (Warning: Strong language)

**5. If U and V are open (resp. closed) then $U \cup V$ is open (resp. $U \cap V$ is closed). If $\{U_i\}_{i=1}^\infty$ is a countable collection of open sets, must $\bigcap_{i \in I} U_i$ be open? Provide a proof or counterexample. Similarly, if $\{A_i\}_{i \in I}$ is an infinite collection of closed sets, must $\bigcap_{i \in I} A_i$ be closed?

*6. Prove that the following sets are open

(a) $\mathbb{R}^n$

(b) $B(r, x)$

(c) $\{(x, y) \in \mathbb{R}^2 \,|\, x > 0\}$

(d) $\{(x, y) \in \mathbb{R}^2 \,|\, x > 1 \text{ and } y > 0\}$ (Hint: You could do this by brute force, or notice that the set can be written as an intersection of two other sets)

(e) $\{(x, y) \in \mathbb{R}^2 \,|\, x \notin \mathbb{Z}\}$ (This one is a bit harder)

*7. Determine and prove whether the following sets are open, closed, or neither open nor closed

(a) $\{(x,y) \in \mathbb{R}^2 \,|\, x \in \mathbb{Z}\}$

(b) $\{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}$

(c) $\{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$

(d) $\{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1, (x,y) \neq (0,0)\}$

(e) $\{(x,y) \in \mathbb{R}^2 \,|\, x > 0, y = \sin(1/x)\}$ (Hint: Plot this beast, then look at what what happens as you approach $x = 0$ along lines $y = C$)

(f) $\{(x,y) \in \mathbb{R}^2 \,|\, 0 < x < 1, 0 < y < 1, x, y \in \mathbb{Q}\}$ (Hint: You might need to wait until we study completeness to answer this)

(g) $\{(x,y) \in \mathbb{R}^2 \,|\, y > x^2\}$

*8. If S is not closed, then there exists $x \in \overline{S}$ but $x \notin S$

**9. If $S \subseteq \mathbb{R}^n$ is not closed, then there exists a sequence $\{x_k\}_{k=1}^{\infty} \subseteq S$ such that no subsequence of $\{x_k\}$ converges in S.

**10. Show that $B(r,x) \subseteq B(r + |x|, 0)$ - i.e. the definition of a bounded set did not depend on where we centre our ball.

***11. Construct an open set which contains the rational numbers $\mathbb{Q}$, but which is a proper subset of $\mathbb{R}$.

***12. Prove that $\overline{S} = \bigcap_{A \supseteq S,\ A \text{ closed}} A$ - i.e. the closure of S is the intersection of all the closed sets containing S.

***13. The intent of this exercise is to show that if we were to start the course over and use open squares instead of open balls in defining open sets, we would have actually had the same definition! Define a map

$$||\cdot||_\infty : \mathbb{R}^n \to \mathbb{R}$$

$$||x||_\infty = \max_i \{|x_i|\}$$

Where $x = (x_1, \ldots, x_i, \ldots, x_n)$. For any $a \in \mathbb{R}^n$, let $S(a, \epsilon) = \{x \in \mathbb{R}^n \,|\, ||x - a||_\infty < \epsilon\}$. We say that a set U is *S-open* (the S stands for square) if and only if for every $a \in U$, there exists $\epsilon > 0$ such that $S(a, \epsilon) \subseteq U$.

(a) Make a sketch of $S(0,1)$ when $n = 2$.

(b) Show that for any $x \in \mathbb{R}^n$, $||x||_\infty \leq ||x|| \leq \sqrt{n}||x||_\infty$, where $||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$

(c) Prove that $U \subseteq \mathbb{R}^n$ is S-open if and only if U is open.

(d) Consider the functions $||\cdot||_p : \mathbb{R}^n \to \mathbb{R}$, $||x||_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$, where $1 < p < \infty$. Plot the sets (by hand, or using a computer) $\{x \in \mathbb{R}^2 \,|\, ||x||_p < 1\}$. Do you expect the p-balls to define the same collection of open sets as the 2-balls? Explain (proof is not necessary).

3 Sequences

Definitions: Bounded, Cauchy, complete, monotone

*1. Let $\{x_k\}$ and $\{y_k\}$ be sequences such that $x_k \to a$ and $y_k \to b$. Show that $x_k + y_k \to a + b$ and $x_k y_k \to ab$.

*2. Let $x_k = \frac{3k+4}{k-5}$. Given $\epsilon > 0$, find an integer K such that $|x_k - 3| < \epsilon$ for all $k > K$.

*3. Let $\{x_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^n$ be a sequence, and let the components be given by $x_k = (x_k^1, x_k^2, \ldots, x_k^n)$. Prove that $x_k \to x = (x^1, \ldots, x^n)$ if and only if $x_k^i \to x^i$ for all i. In other words, a sequence in $\mathbb{R}^n$ converges if and only if each of its components converges.

**4. Find an example of a sequence $\{x_k\}$ such that $|x_{k+1} - x_k| \to 0$, but $\{x_k\}$ is not Cauchy.

**5. Let $S \subseteq \mathbb{R}$, and set $L = \inf S$. Show there exists a sequence $\{x_k\}$ converging to L.

**6. A sequence $\{x_k\}_{k=1}^{\infty}$ converges to x if and only if for every subsequence $\{x_{k_n}\}_{n=1}^{\infty}$, $x_{k_n} \to x$.

*7. Prove whether each of the following sequences converges or does not have a limit

(a) $((-1)^k, 0)$

(b) $(\cos(\pi k/2), \sin(\pi k/2))$

(c) $\left(\frac{\cos \pi k}{k}, \frac{\sin \pi k}{k}\right)$

(d) $((1 + 1/k)\cos(\pi k/2), (1 + 1/k)\sin(\pi k/2))$

(e) $(\cos(Ck), \sin(Ck))$ where C/π is irrational

(f) $(k^2, 0, 1/k)$

**8. If $\{x_k\} \subseteq \mathbb{R}^n$ is a Cauchy sequence and there exists a subsequence $\{x_{k_n}\}$ such that $x_{k_n} \to x$, then $x_k \to x$. (c.f. the previous question! Cauchy sequences behave much better)

***9. Construct a sequence $\{z_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^2$ with the property that for any point $z = (x, y)$ such that $x^2 + y^2 = 1$, there exists a subsequence $\{z_{k_n}\}_{n=1}^{\infty}$ such that $z_{k_n} \to z$.

***10. Construct a sequence $\{x_k\} \subseteq \mathbb{R}^2$ with the property that for any $x \in \mathbb{R}^2$, there exists a subsequence $\{x_{k_n}\}$ which converges to x. (Obviously this solves the previous exercise as well)

†11. We know that $\mathbb{Q} \subseteq \mathbb{R}$ is not open, by the completeness axiom. The next best thing we might ask is whether $\mathbb{Q}$ can be written as a countable intersection of open sets (which may not be open).

We say that a subset $S \subseteq \mathbb{R}^n$ is *dense* if and only if every non empty open set U has $U \cap S \neq \emptyset$.

Theorem 1. *If $\{U_i\}_{i=1}^{\infty}$ is a collection of open and dense subsets of $\mathbb{R}^n$, then $\bigcap_{i=1}^{\infty} U_i$ is dense.*

(a) Show that any open ball $B(x, r)$ contains a closed closed ball, and that any closed ball $\overline{B(x, r)}$ contains an open ball.

(b) Show that if S is dense and W is open, then $S \cap W$ is dense in W (i.e. any open set V such that $V \cap W \neq \emptyset$ has $S \cap V \cap W \neq \emptyset$.

(c) Let W be any open set. Construct a Cauchy sequence $\{x_k\}_{k=1}^{\infty}$ such that $x_k \in W$ for all k, and $x_k \in U_i$ whenever $k > i$.

(d) Conclude that $W \cap \bigcap_{i=1}^{\infty} U_i \neq \emptyset$.

Corollary 1. *There cannot exist a collection of open sets $\{U_k\}$ such that $\mathbb{Q} = \bigcap_{k=1}^{\infty} U_k$.*

(a) If V is dense and $V \subseteq U$, then U is dense.

(b) If U is dense and $p \in \mathbb{R}^n$ is any point, then $U \setminus \{p\}$ is dense.

(c) Prove the corollary.

4 Continuity

Definitions: Continuous function, uniform continuity, sequential continuity

*1. Find the limit, if it exists, or prove that the limit does not exist

(a) $\lim_{(x,y)\to(0,0)} \left(5x^3 - x^2y^2\right)$

(b) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ This one is a bit harder. Hint: Bound the Euclidian norm using the sup norm.

(c) $\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$.

(d) $\lim_{(x,y)\to(0,0)} \frac{x^2+y}{\sqrt{x^2+y^2}}$

(e) $\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{x^2+2y^2}$. This one is also hard. Try and bound $\sin(y)$ by a function which is easier to understand.

**2. Define a function $f : \mathbb{R}^2 \setminus \{(x,y) \mid x = 0\} \to \mathbb{R}$ as follows:

$$f(x,y) = \frac{\sin(xy)}{x}$$

How should you define $f(x,y)$ at $x = 0$ so that $f(x,y)$ extends to a continuous function on all of $\mathbb{R}^2$?

**3. Prove the following are equivalent:

(a) $f : \mathbb{R}^m \to \mathbb{R}^n$ is ϵ-δ continuous

(b) For every $V \subseteq \mathbb{R}^n$ open, $f^{-1}(V)$ is open

(c) For every $V \subseteq \mathbb{R}^n$ open, $\forall x \in \mathbb{R}^m$, if $f(x) \in V$ then there exists an open set $U \subseteq \mathbb{R}^m$ containing x such that $f(U) \subseteq V$.

*4. Find an example of a continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ and an open set $U \subseteq \mathbb{R}^n$ such that $f(U)$ is not open. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ and there exists a continuous $g : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(g(x)) = x$ and $g(f(x)) = x$. If U is open, must $f(U)$ be open? Must $g(U)$ be open?

**5. If $f : S \to \mathbb{R}$ is uniformly continuous and $\{x_k\}$ is a Cauchy sequence, then $\{f(x_k)\}$ is Cauchy

***6. If $f : S \to \mathbb{R}$ is uniformly continuous, then there exists a unique continuous function $\tilde{f} : \overline{S} \to \mathbb{R}$ such that $\tilde{f}\big|_S = f$

***7. Let $f, g : \mathbb{R}^n \to \mathbb{R}^k$ be continuous functions and suppose that $D \subseteq \mathbb{R}^n$ is a dense set. If $f(x) = g(x)$ for every $x \in D$, then $f(x) = g(x)$ for every $x \in \mathbb{R}^n$.

***8. Let $C([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous on } [0,1]\}$ be the set of all continuous functions on the closed unit interval. Define the following function:

$$d : C([0,1]) \times C([0,1]) \to \mathbb{R}$$

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$$

Show that d has the following properties:

(a) $d(f,g) \geq 0$, and $d(f,g) = 0$ if and only if $f(x) = g(x)$ for every $x \in [0,1]$

(b) $d(f,h) \leq d(f,g) + d(g,h) \quad \forall\, f,g,h \in C([0,1])$

(c) $d(f,g) = d(g,f) \quad \forall\, f,g \in C([0,1])$

We say that d defines a *metric* on $C([0,1])$. We can use d to make sense of open sets in $C([0,1])$. Fix a continuous function $g \in C([0,1])$, and let $U = \{f \in C([0,1]) \,|\, f(x) < g(x) \,\forall\, x \in [0,1]\}$. Show that U is an open set with respect to the metric d.

Is U still open if instead of $C([0,1])$, we had started with $C(\mathbb{R})$, continuous functions on $\mathbb{R}$, and set $U = \{f \in C(\mathbb{R}) \,|\, f(x) < g(x) \,\forall\, x \in \mathbb{R}\}$?

5 Compactess

*1. Use the Bolzano-Weierstrass theorem to prove that if $f : \mathbb{R}^n \to \mathbb{R}^k$ is continuous and $K \subseteq \mathbb{R}^n$ is compact, then $f(K)$ is compact.

*2. Use the Bolzano-Weierstrass theorem to prove that if $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ are compact sets, then $A \times B = \{(x,y) \in \mathbb{R}^{n+m} \,|\, x \in A, y \in B\}$ is compact.

**3. Suppose that $f : S \to \mathbb{R}$ is continuous, and let $L = \inf\{f(x) \mid x \in S\}$. Can we always guarantee that there exists $y \in S$ such that $f(y) = L$? Prove the statement, or else provide a counterexample. If the statement is false, how could we have modified the problem to make it true?

**4. The distance between two sets $U, V \subseteq \mathbb{R}^n$ is defined by:

$$d(U,V) = \inf\{|x-y| : x \in U, y \in V\}$$

(a) Show that $d(U,V) = 0$ if either $\exists\, x \in \overline{U} \cap V$ or $\exists\, x \in \overline{V} \cap U$

(b) Show that if U is compact, V is closed, and $U \cap V = \emptyset$, then $d(U,V) > 0$.

(c) Show that the compactness of U in the previous part was necesssary by giving an example of two closed sets U and V in $\mathbb{R}^2$ which share no point in common, but satisfy $d(U,V) = 0$.

**5. If $f : K \to \mathbb{R}$ is continuous and K is compact then f is uniformly continuous

***6. Use the Bolzano-Weierstrass theorem to prove that if $K_1 \supset K_2 \supset K_3 \supset K_4 \supset \ldots$ is a chain of proper containments and each $K_i \subseteq \mathbb{R}^n$ is compact, then $\bigcap_{i=1}^{\infty} K_i \neq \emptyset$

***7. Consider the collection $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ and define a function:

$$d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$$

$$d(n,m) = \begin{cases} 1 & n \neq m \\ 0 & n = m \end{cases}$$

Check that d satisfies the following properties:

(a) $d(m,n) = 0$ if and only if $m = n$

(b) $d(l,n) \leq d(l,m) + d(l,n) \quad \forall\, l,m,n \in \mathbb{Z}$

(c) $d(m,n) = d(n,m) \quad \forall\, n,m \in \mathbb{Z}$

(d) $d(m,n) \geq 0 \quad \forall\, n,m \in \mathbb{Z}$

We say that d defines a *metric* on $\mathbb{Z}$ (c.f. the Euclidian norm on $\mathbb{R}$, $d(x,y) = \sqrt{x^2+y^2}$). Prove that with respect to this metric, $\mathbb{Z}$ is closed and bounded. Construct a sequence $\{n_i\} \subseteq \mathbb{Z}$ which has no convergent subsequence.

Show that $\exists \epsilon > 0$ such that $\forall\ N \in \mathbb{N}$ and any collection of points $\{n_i\}_{i=1}^N \subseteq \mathbb{Z}$, we have $\mathbb{Z} \not\subseteq \bigcup_{i=1}^N B(\epsilon, n_i)$. In this case, we say that $\mathbb{Z}$ is not *totally bounded* with respect to the metric d.

***8. Let $U \subseteq \mathbb{R}^n$ be an open set. We say that $f : U \to \mathbb{R}$ is *proper* if and only if f is continuous, and for every compact set $K \subseteq \mathbb{R}$, $f^{-1}(K) \subseteq U$ is compact. Suppose that we have a finite set $A \subseteq \mathbb{R}^n$ and a function $f : \mathbb{R}^n \backslash A \to \mathbb{R}$ such that for every $a \in A$,

$$\lim_{x \to a} |f(x)| = \lim_{x \to \infty} |f(x)| = \infty$$

Show that f is proper.

6 Connectedness

Definitions: Convex set, connected set, path connected set

*1. Show directly from the definition that the following sets are disconnected:

(a) The hyperbola $\left\{(x,y) \in \mathbb{R}^2 \,|\, x^2 - y^2 = 1\right\}$

(b) Any finite set of points in $\mathbb{R}^n$ with more than two elements

(c) $\left\{(x,y,z) \in \mathbb{R}^3 \,|\, xyz > 0\right\}$

**2. Let $\{S_n\}_{n=1}^\infty$ be a collection of connected sets with the property that $S_n \cap S_{n+1} \neq \emptyset$ for every n. Prove that $\bigcup_{n=1}^\infty S_n$ is connected. (Hint: This is easier if you first try it for only two sets)

*3. If A and B are connected, must $A \cup B$ be connected? Must $A \cap B$ be connected? Provide a proof or a counterexample in both cases. If A and B are convex, is $A \cap B$ convex?

**4. Show that for any $\epsilon > 0$ and any $x \in \mathbb{R}^n$, $B(\epsilon, x)$ is a convex set.

**5. Show that if S is connected, then $\overline{S}$ is connected. (Hint: I think the contrapositive is easier to prove)

**6. Show that $\left\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 = 1\right\}$ is connected.

***7. Let $S \subseteq \mathbb{R}^n$. S is disconnected if and only if there exists a continuous function $f : S \to \mathbb{R}$ such that $f(S) = \{0,1\}$.

7 Parameterized curves and surfaces

*1. Sketch the following parameterized curves, indicating the direction of motion using arrows:

(a) $(x(t), y(t)) = (R\sin(t), R\cos(t))$ for fixed $R > 0$, and $t \in [0, 2\pi)$

(b) $(x(t), y(t), z(t)) = (t+2, 3, 2t-1)$ for $t \in [-1, 1]$

(c) $(x(t), y(t), z(t)) = (\cos(t), \sin(t), t)$ for $t \in [-\pi, 4\pi]$

(d) $(x(t), y(t), z(t)) = (t, t\cos(t), t\sin(t))$ for $t \in [0, 6\pi]$

*2. Sketch the following surfaces. When a constant c is indicated, investigate the cases when $c = -1$, $c = 0$, and $c = 4$.

(a) $\{(x, y, z) \in \mathbb{R}^2 \mid x + y + z = 1\}$

(b) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = c\}$

(c) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = c\}$

(d) $\{(x, y, z) \in \mathbb{R}^3 \mid -x^2 + y^2 + z^2 = 1\}$

(e) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 - z^2 = c\}$

(f) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + z^2 = 4\}$

(g) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0\}$

*3. Find parameterizations $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ of the regions of all the above surfaces where $z > 0$. In particular, you should find the open set U on which F is defined.

*4. Sketch the region in $\mathbb{R}^3$ bounded by the surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$.

*5. Sketch the region in $\mathbb{R}^3$ bounded by the surfaces $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 = 1$ for $1 \leq z \leq 2$.

**6. Let C be the curve obtained by intersecting the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x + z = 1$. Find a map $\gamma : I \to \mathbb{R}^3$ parameterizing the curve C.

**7. Sketch the surface $z = (x^2 + y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$. If you use a computer, you're cheating!

**8. Let $S \subseteq \mathbb{R}^3$ be the set of points, such that each point in the set has equal distance to $(-1, 0, 0)$ and to the plane $x = 1$. Find an equation $F(x, y, z)$ such that $S = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$. Sketch this surface.

8 Differentiability

*1. Compute the partial derivatives $\partial_i f$, and the total derivative Df for each of the following functions $f : \mathbb{R}^n \to \mathbb{R}^k$:

(a) $f(x, y, z) = x^y$

(b) $f(x, y) = x^y$

(c) $f(x, y, z) = (x^y, z)$

(d) $f(x, y) = \sin(x \sin(y))$

(e) $f(x, y, z) = (x + y)^z$

(f) $f(x, y, z) = (\log(x^2 + y^2 + z^2), xyz)$

(g) $f(x, y) = \sin(xy)$

*2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $f(x, y) = (x^2 - y^2, 2xy)$.

(a) Calculate Df and $\det Df$

(b) Let $S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1, x \geq 0, y \geq 0\}$. Make a sketch of $f(S)$ by showing where some of the coordinate curves get mapped.

$\star$3. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^2$ is a function such that $f(0,0,0) = (1,2)$ and:

$$Df_{(0,0,0)} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $g(x,y) = (x+2y+1, 3xy)$. Find $D(F \circ G)_{(0,0,0)}$

$\star$4. Find an equation for the tangent plane T_pS to the following surfaces at the indicated point

(a) $S = \{(x,y,z) \mid x^2+2y^2+3z^2 = 6\}$ at $(1,1,-1)$.

(b) $S = \{(x,y,z) \mid xyz^2 - \log(z-1) = 8\}$ at $(-2,-1,2)$.

(c) $S = \{(x,y,z) \mid x^2+y^2 = 1\}$ at $(1/\sqrt{2}, 1/\sqrt{2}, 1)$

$\star$5. Suppose that $f(x,y,z,t)$, $x(t)$, $y(x,t,s)$, and $z(y,x)$. Use the chain rule to find an expression for $\frac{\partial f}{\partial t}$ and $\frac{\partial f}{\partial s}$.

$\star\star$6. Show that $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sqrt{|xy|}$ is not differentiable at $(x,y) = (0,0)$.

$\star\star$7. Let $F(\varphi,\theta) = (x(\varphi,\theta), y(\varphi,\theta), z(\varphi,\theta))$ be the spherical polar coordinate system on the unit sphere $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$.

(a) Sketch the coordinate curves $\theta = \pi/4$ and $\varphi = \pi/2$

(b) Compute the derivative to the coordinate curves from part (1) at the point $(\varphi,\theta) = (\pi/2, \pi/4)$. Add these arrows to your plot.

(c) Prove that $\partial_\varphi F \times \partial_\theta F$ is parallel to $\nabla(x^2+y^2+z^2)$.

$\star\star\star$8. Let $f : \mathbb{R}^3 \to \mathbb{R}$, and suppose that $\nabla f \neq 0$. Prove that if $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable parameterization of a level set $S = f^{-1}(c)$, then $\exists\, \lambda \in \mathbb{R}$ such that $\partial_u F \times \partial_v F = \lambda \nabla f$.

$\star\star$9. Let $f, g : \mathbb{R}^n \to \mathbb{R}^m$. If f and g are differentiable at $x \in \mathbb{R}^n$, then $D(f+g)_x = Df_x + Dg_x$ and $D(fg)_x = f(x)Dg_x + g(x)Df_x$. Notice these are generalizations of the sum and product rules for differentiation from last year. (Hint: Notice that the maps $(x,y) \to x+y$ and $(x,y) \mapsto xy$ are themselves differentiable maps)

$\star\star$10. If $f : S \to \mathbb{R}$ is differentiable on an open, connected set and $\nabla f(x) = 0$ for all $x \in S$, then f is constant. (Hint: First solve the problem for a ball $B(\epsilon, x)$, then for the general case try and imitate the proof that every open and connected set is path connected)

$\star\star\star$11. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable, where U is star convex about a point $p \in U$, and $f(p) = 0$. Show that $\exists\, g_1, \ldots, g_n : U \to \mathbb{R}$ such that $f(x) = \sum_{i=1}^n x_i g_i(x)$. (Hint: May as well assume that U contains the origin and $p = 0$. Consider the function $f(tx)$)

While this problem seems like an uninteresting result, it is actually quite useful. It is a main technical lemma involved in proving "the Morse lemma", which is a powerful result which gives insight into the structure of manifolds. It is also inherently interesting as an algebraic result in itself; to those who know about "rings", the result above says that the ideal of differentiable functions which vanish at a point is generated by the functions $x_1, \ldots, x_n$.

***12. This question looks hard, but is actually very easy. Let $C([0,1])$ be the collection of all continuous functions on the closed unit interval, given the sup norm as above. Consider the map:

$$F = \int_0^t : C([0,1]) \to C([0,1])$$

$$f \mapsto \int_0^t f(x)\,dx$$

Compute DF_f. (Hints: What quantity do you need to estimate when computing a derivative? Think about the properties of integration. Also, $C([0,1])$ is a vector space over $\mathbb{R}$!)

9 Taylor's theorem

Facts to know: Taylor series of e^x, $\cos(x)$,$\sin(x)$,$1/(1-x)$, $\log(1-x)$ based at $x=0$.

*1. Let $f(x,y) = e^x\cos(y)$. Verify by hand that $\partial_x\partial_y f = \partial_y\partial_x f$.

**2. Derive the following version of the "product rule" for partial derivatives; if α is any multi-index, then:

$$\partial^\alpha(fg) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}\partial^\beta f \partial^\gamma g$$

**3. Prove the following version of the binomial theorem: For any pair of points $x,y\in\mathbb{R}^n$, $(x+y)^\alpha = \sum_{\beta+\gamma=\alpha}\frac{\alpha!}{\beta!\gamma!}x^\beta y^\gamma$

*4. Find the 3rd order Taylor polynomial of the following functions:

- $f(x,y) = \sin(x)\cos(y)$ based at the point $(0,0)$.
- $f(x,y) = \frac{1}{1+x-y}$ based at the point $(0,0)$.
- $f(x,y) = \log(1+x-y)$ based at the point $(0,0)$.
- $f(x,y) = x + \cos(\pi y) + x\log y$ based at the point $(3,1)$
- $f(x,y,z) = x^2y + z$ based at $(1,2,1)$. Why should the remainder be zero?

†5. For every n, let $f_n : U \to \mathbb{R}$ be a $C^k(U)$ function, where $U \subseteq \mathbb{R}^n$ is open. We say that the sequence of functions $\{f_n\}_{n=1}^\infty$ converges uniformly to a limit f on $K \subseteq U$ if and only if $\lim_{n\to\infty}\sup_{x\in K}|f(x) - f_n(x)| = 0$. If $\{f_n\}$ converges uniformly to f, we write $f_n \to f$ uniformly.

(a) Show that if $f_n : S \to \mathbb{R}$ are continuous and $f_n \to f$ uniformly, then f is continuous

(b) Conclude that if $\partial_\alpha f_n$ converge uniformly to $\partial_\alpha f$ for every multi-index α with $|\alpha| \le k$, then $f \in C^k(U)$.

(c) Prove that if $K \subseteq U$ is compact and $f \in C^k(U)$, then the Taylor polynomials $P_n(x)$ converge to f uniformly on K.

The result of our work is the following theorem:

Theorem 2. *Polynomial functions are dense in $C^k(U)$, using the topology of uniform convergence on compact sets.*

10 Optimization

*1. Find all the critical points of the following functions. Say whether the critical points are local maxima, local minima, saddle points, or otherwise.

(a) $f(x,y) = x^4 - 2x^2 + y^3 - 6y$

(b) $f(x,y) = (x-1)(x^2 - y^2)$

(c) $f(x,y) = x^2y^2(1-x-y)$

(d) $f(x,y) = (2x^2 + y^2)e^{-x^2-y^2}$

(e) $f(x,y,z) = xyz(4-x-y-z)$

*2. Find the extreme values of $f(x,y,z) = x^2 + 2y^2 + 3z^2$ on the unit sphere, $x^2 + y^2 + z^2 = 1$.

**3. What conditions on a, b, and c guarantee that $f(x,y) = ax^2 + bxy + cy^2$ has local max, a local min, or a saddle point at $(0,0)$?

**4. What is the volume of the largest box which can be fit inside of the sphere $x^2 + y^2 + z^2 = 1$?

**5. Use the method of Lagrange multipliers to find the smallest distance between the parabola $y = x^2$ and the line $y = x - 1$.

***6. Let $f(x,y) = (y - x^2)(y - 2x^2)$. Show that the origin is a degenerate critical point of f. Prove that f restricted to any line through the origin has a local minimum, but f does not have a local minimum at the origin. (Hint: Consider the regions where $f > 0$ and $f < 0$).

**7. Find the minimum of the function $f(x,y,z) = x^2 + y^2 + z^2$ on the surface $x^2 + y^2 - z^2 = c$, where $c \in \mathbb{R}$. Hint: You'll need to break this into cases, depending on the value of c.

***8. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Show that the on the set $S = \{v \in \mathbb{R}^n \mid ||v|| = 1\}$, the maximum and minimum of A are the largest and smallest eigenvalues of A, respectively.

11 Manifolds in $\mathbb{R}^n$

11.1 The Implicit and Inverse Function Theorems

Theorem Statements: The Inverse Function Theorem, The Implicit Function Theorem

*1. Determine whether the equation $\sin(xyz) = z$ may be solved for z as a function of x and y in a neighbourhood of the point $(x,y,z) = (\pi/2, 1, 1)$.

*2. Find conditions on x and y which guarantee that one can locally solve the following for $u(x,y)$ and $v(x,y)$:

$$\begin{aligned} xu^2 + yv^2 &= 9 \\ xv^2 - yu^2 &= 7. \end{aligned}$$

*3. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x,y,z) = \left(ye^x + \sin(\pi y)\cos(z),\ \cos(yz),\ z^2\right).$$

Determine whether f is invertible in a neighbourhood of $(0, 1, \pi/2)$.

**4. Show that the following system always has a solution for sufficiently small a,

$$\begin{aligned} x + y + \sin(xy) &= a \\ \sin(x^2 + y) &= 2a \end{aligned}$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant C^1 function such that $f'(0) \neq 0$ and $f(x+y) = f(x)f(y)$. Define $F : \mathbb{R}^2 \to \mathbb{R}$ by $F(x, y) = f(x)f(y)$. Determine what conditions (if any) must be imposed upon y to ensure that y can be solved as a function of x on the set $\{(x, y) : F(x, y) = 1\}$. **Bonus: Write down an explicit formula for y as a function of x.

**6. Define the set

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

to be the set of 2×2 matrices. Define a map $g : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ by $g(A) = A^2$. Determine whether g is invertible in a neighbourhood of $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

***7. In this problem, we will show that for a special class of polynomials, slightly perturbing the coefficients will preserve the number of roots of the equation.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a degree n *polynomial*. Show that if all the roots of f are distinct, then for any root r we necessarily have $f'(r) \neq 0$. [Hint: We say that a root r has multiplicity k if $f(x) = (x - r)^k q(x)$ and $q(r) \neq 0$. All the roots of a polynomial are distinct if every root has multiplicity 1].

(b) For fixed $(c_0, \ldots, c_{n-1}$ let $f(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be a function with distinct roots. Identify the $(c_0, \ldots, c_{n-1})$ with a point in $\mathbb{R}^n$. Show that for each root r, there is a neighbourhood of U_r of $(c_0, c_1, \ldots, c_{n-1}) \in \mathbb{R}^n$ and a neighborhood V_r of r such that if $(d_0, \ldots, d_{n-1}) \in U_r$ then $x^n + d_{n-1}x^{n-1} + \cdots + d_1 x + d_0$ has a root in V_r.

(c) Use part b to conclude that a degree n polynomial with exactly $m < n$ roots, all of which are distinct, has the same number of roots under small perturbation of its coefficients.

(d) Does this result still hold if the roots are no longer distinct? Prove the result or give a counter-example.

***8. A map $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is said to be open if whenever U is open then $\mathbf{f}(U)$ is open. Show that if $\mathbf{f}$ is C^1 and $D\mathbf{f}(\mathbf{x}_0)$ is invertible for all $\mathbf{x}_0 \in \mathbb{R}^n$ then $\mathbf{f}$ is an open map.

***9. Let $M_2(\mathbb{R})$ be defined as in Problem (6) and define the subset

$$O_2(\mathbb{R}) = \left\{ X \in M_2(\mathbb{R}) : XX^T = \mathrm{Id} \right\}.$$

Show that $O_2(\mathbb{R})$ defines a C^1-manifold of $M_2(\mathbb{R})$ and determine its dimension.

†10. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if for every $t \in [0, 1]$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$f(t\mathbf{x} + (1 - t)\mathbf{y}) < tf(\mathbf{x}) + (1 - t)f(\mathbf{y}).$$

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function which is strictly convex, non-negative, and satisfies $F(0,0) = 0$. Show that there is an *everywhere concave-down* function $f : \mathbb{R} \to \mathbb{R}$, such that in a neighbourhood of $(0,1)$, $F(x, f(x)) = F(0,1)$.

11.2 Curves and Surfaces

*1. For each of the following manifolds, determine which can be written as the graph of a function, the zero-locus of a function, and parametrically. Give the appropriate functions in each case.

(a) The ellipse $ax^2 + by^2 = c^2$,

(b) The set $\left\{(t^2 + t, 2t - 1) : t \in \mathbb{R}\right\}$,

(c) The plane $3x - 4y + 3z = 10$,

(d) The sphere of radius r, $S_r = \left\{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = r\right\}$,

(e) The cylinder $\left\{(x, y, z) : x^2 + y^2 = 4\right\}$,

(f) The intersection of the plane $x + z = 1$ with the sphere $x^2 + y^2 + z^2 = 1$,

(g) If $f : [a, b] \to \mathbb{R}$ let S be the space defined by revolving the graph of f about the x-axis.

*2. Determine whether the following spaces are smooth:

(a) The set $C = \{(\cos(t), \sin(2t) : t \in [0, 2\pi)\}$.

(b) Let S be the surface defined by the image of $g : \mathbb{R}^2 \to \mathbb{R}^3$, $g(s,t) = (3s, s^2 - 2t, s^3 + t^2)$.

(c) Let $S = F^{-1}(0)$ where $F(x, y, z) = 3xy + x^2 + z$.

(d) Let $S = F^{-1}(0)$ where $F(x, y, z) = \cos(xy) + e^z$

**3. Let $U = \{(x, y, z) : x > 0, y > 0, z > 0\}$ be the first octant, and let $g : U \to \mathbb{R}^4$ be given by

$$g(t, u, v) = (tu, tv, uv, tuv).$$

Determine whether the image of g defines a smooth surface.

**4. For what values of c do the following equations define a C^1 surface?

(a) $x^2 + y^2 + z^2 = c_1$, $x^2 + y^2 - z^2 = c_2$,

(b) $xyz = c$

**5. Let $F_1, F_2 : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions and let $F_3(\mathbf{x}) = F_1(\mathbf{x})F_2(\mathbf{x})$. If $S_i = \{\mathbf{x} \in \mathbb{R}^n : F_i(\mathbf{x}) = 0\}$ show that $S_3 = S_1 \cup S_2$. [In particular, this shows that it when analyzing the zero-locus of a function which is the product of two functions, it is sufficient to look at each constituent function separately.]

**6. Consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ give by $\gamma(t) = (2e^{-t/2}\cos(t), 2e^{-t/2}\sin(t))$.

(a) Show that $\gamma(t)$ defines a C^1 curve.

(b) Calculate the speed of this curve as a function of t. [Hint: The velocity is $\gamma'(t)$ so the speed is $\|\gamma'(t)\|$]

(c) We define the unit tangent vector to the curve to be $T(t) = \gamma'(t)/\|\gamma'(t)\|$. Compute the unit tangent vector.

(d) We will see later in the course that the arc-length of a curve on the interval $[0,t]$ is given by

$$s(t) = \int_0^t \|\gamma'(u)\| du.$$

Compute the arc-length function $s(t)$ for the curve γ.

(e) Inverting the arc-length formula gives a function $t(s)$ (time as a function of arc-length). The reparameterization of the curve $\gamma(t)$ using $t = t(s)$ is known as the *arclength parameterization.* Compute the arc-length parameterization of $\gamma(t)$; that is, compute $\gamma(t(s))$.

12 Integration of $\mathbb{R}$-valued functions

Theorem Statements: Fubini's Theorem Definitions: Jordan measure, measure zero

12.1 Integration on the Line

**1. Using any equivalent definition of integration (see Question 4), show that integration is a linear operator; that, show that if f_1 and f_1 are integrable on $[a,b]$, then $c_1 f_1 + c_2 f_2$ is integrable on $[a,b]$ and

$$\int_a^b \left[c_1 f_1(x) + c_2 f_2(x)\right] dx = c_1 \int_a^b f_1(x) dx + c_2 \int_a^b f_2(x) dx.$$

**2. Compute the Jordan measure of the set $S = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \subseteq \mathbb{R}$. Justify your answer rigorously. [Hint: To show that a set has measure 0, show that for any ϵ you can always find a covering whose total length is less than ϵ].

**3. More generally, show that if $(a_n)_{n=1}^{\infty}$ is any convergent sequence, then (a_n) has measure zero.

***4. Prove that the following statements are equivalent:

(a) The bounded function $f : [a,b] \to \mathbb{R}$ satisfies the following: there is a real number I such that, for all $\epsilon > 0$ there exists a $\delta > 0$ having the property that whenever $P \in \mathcal{P}_{[a,b]}$ is a partition of $[a,b]$ with $\ell(P) < \delta$, then

$$|S(f,P) - I| < \epsilon.$$

(b) If $f : [a,b] \to \mathbb{R}$ is a bounded function, then

$$\sup_{P \in \mathcal{P}_{[a,b]}} L_f(P) = \inf_{P \in \mathcal{P}_{[a,b]}} U_f(P).$$

(c) The bounded function $f : [a,b] \to \mathbb{R}$ satisfies the following property: For every $\epsilon > 0$ there exists a partition $P \in \mathcal{P}_{[a,b]}$ such that

$$U_f(P) - L_f(P) < \epsilon.$$

(d) The bounded function $f : [a,b] \to \mathbb{R}$ satisfies the property that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $P, Q \in \mathcal{P}_{[a,b]}$ are two partitions satisfying $\ell(P) < \delta, \ell(Q) < \delta$ then

$$|S(f,P) - S(f,Q)| < \epsilon.$$

[Hint: For the [(b) $\Rightarrow$ (a)] case, let $\epsilon_n = \frac{1}{2^n}$ and (δ_n) be the corresponding δ's guaranteed to exist by (b). Use these to create a Cauchy sequence x_n and then use completeness of $\mathbb{R}$.]

***5. Show that every Riemann integrable function $f : [a, b] \to \mathbb{R}$ is bounded; that is, there exists some $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.

***6. Show that if $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then $F(x) = \int_a^x f(s)ds$ is uniformly continuous on $[a, b]$. [Hint: Use the fact that integrable functions are bounded, together with the Squeeze Theorem.]

12.2 Multivariable Integration

***1. Let $f : [a, b] \to \mathbb{R}$ be a bounded, integrable function.

(a) Show that the graph of f, $\Gamma(f) := \{(x, f(x)) : x \in [a, b]\} \subseteq \mathbb{R}^2$ has zero content.

(b) If f is non-negative, show that $S = \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$ is measurable, and $m(S) = \int_a^b f(x)dx$.

**2. Let Z_i, $i = 1, \ldots, n$ be a collection of zero measure sets. Show that the union $\bigcup Z_i$ also has zero measure.

***3. Show that if $f : \mathbb{R} \to \mathbb{R}^2$ is a C^1 function, then for any interval $I \subseteq \mathbb{R}$, $f(I)$ has zero Jordan measure.

***4. If $S = \{x_1, \ldots, x_n\}$ is a finite set consisting of precisely n-elements, show that S has zero Jordan measure.

***5. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. If $g : [a, b] \to \mathbb{R}$ is another function and $S = \{x : f(x) \neq g(x)\}$ contains exactly n-points, show that g is also Riemann integrable. [Note: You must prove this from scratch. If you wish to invoke a corollary or result from class, you must first prove it.]

12.3 Iterated Integrals

*1. Determine the integral of each function on the specified rectangle:

(a) $f(x, y) = e^x \cos(y)$ for $0 \leq x \leq 1$ and $\frac{\pi}{2} \leq y \leq \pi$,

(b) $f(x, y) = e^{x-y}$ for $0 \leq x \leq 1$ and $-2 \leq y \leq -1$,

(c) $f(x, y) = x^2 6 - 3xy^2$ for $1 \leq x \leq 2$ and $-1 \leq y \leq 1$,

(d) $f(x, y) = \frac{1}{x+y}$ for $0 \leq x \leq 1$ and $1 \leq y \leq 2$,

(e) $f(x, y) = y \cos(xy)$ for $0 \leq x \leq 1$ and $0 \leq \leq \pi$,

(f) $f(x, y) = e^y \sin(xy^{-1})$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $1 \leq y \leq 2$,

(g) $f(x, y) = \sin(x + y)$ for $0 \leq x, y \leq \frac{\pi}{2}$,

(h) $f(x, y) = 4xy\sqrt{x^2 + y^2}$ for $0 \leq x \leq 3$ and $0 \leq y \leq 1$.

*2. In this question we will generalize the notions of even and odd, and show multivariable analogs of single variable results. Let $r > 0$ and set $R = \{(x, y) \in \mathbb{R}^2 : -r \leq x, y \leq r\}$.

(a) Let f be an integrable function such that $f(-x,-y) = -f(x,y)$. Show that

$$\iint_R f(x,y)dxdy = 0.$$

(b) Let f be an integrable function such that $f(x,-y) = -f(x,y)$. Show that

$$\iint_R f(x,y)dxdy = 0.$$

*3. Let R be the region in the xy-plane bounded by the curves $y = 2x$ and $y = x^2$. Determine the area bounded by R and the paraboloid $z = x^2 + y^2$.

*4. Determine the area given by the intersection of the two cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$ for any $r > 0$.

*5. In each case, determine $\iint_S f(x,y)dA$ where f and S are specified:

(a) $f(x,y) = 1 + x + y$, $S = \{0 \leq x \leq 1, 0 \leq y \leq e^x\}$,
(b) $f(x,y) = (x-y)^2$, S is the region bounded between x^2 and x^3,
(c) $f(x,y) = y$, $S = \{x^2 + y^2 \leq 1\} \cap \{x^2 + (y-1)^2 \leq 1\}$,
(d) $f(x,y) = x^2y^2$, $S = \{-y^2 \leq x \leq y^2, 0 \leq y \leq 1\}$,
(e) $f(x,y) = xy$, S the area bounded by the lines $y = x - 1$ and $y^2 = 2x + 6$,
(f) $f(x,y) = 1 + x$, S is the area bounded between $x + y = 0$ and $y + x^2 = 1$,
(g) $f(x,y) = \frac{\sin(y)}{y}$, S is the area bounded between $y = x$ and $y = \sqrt{x}$.

*6. Find the volume of the solid bounded by the surfaces $z = 3x^2 + 3y^2$ and $x^2 + y^2 + z = 4$.

**7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an integrable function, and define

$$G(x) = \int_a^x \int_a^s f(s,t)dtds.$$

Show that one can equivalently write

$$G(x) = \int_a^x \int_t^x f(s,t)dsdt.$$

**8. Let $R = [0,1] \times [0,1]$ and consider the function $f : R \to \mathbb{R}$ given by $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$.

(a) Show that

$$\iint_R f(x,y)dx\ dy \neq \iint_R f(x,y)dy\ dx.$$

(b) Is this a contradiction to Fubini's Theorem? Why or why not?

**9. (a) Let $\alpha \in \mathbb{R}$ be an arbitrary non-zero constant. Compute

$$\int \frac{x-\alpha}{(x+\alpha)^3}dx.$$

[Hint: To integrate $x/(x+\alpha)^3$ make the substitution $u = x + \alpha$]

(b) Let R be the rectangle $R = [0,1] \times [0,1]$ and compute the iterated integrals

$$\iint\limits_R \frac{x-y}{(x+y)^3} dxdy, \qquad \iint\limits_R \frac{x-y}{(x+y)^3} dydx.$$

[Notice that the order of integration is changed!]

(c) You should have found in part (c) that the integrals did not agree. Explain why this is not a contradiction to Fubini's theorem.

**10. Evaluate the integral of the following functions on the specified domain:

(a) $f(x,y,z) = y$ over the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $2x+2y+z = 4$,

(b) $f(x,y,z) = z$ over the region bounded by $y^2 + z^2 = 9$, $x = 0$, $y = 3x$ and $z = 0$ in the first octant.

(c) $f(x,y,z) = 1$ over the region bounded by $y = x^2$, $z = 0$ and $y + z = 1$.

***11. Compute the given interval on the given domain:

(a) $\iint_R \left[2 + x^2y^3 - y^2\sin(x)\right] dA$ where $R = \{|x| + |y| \leq 1\}$,

(b) $\iint_R \left[ax^2 + by^3 + \sqrt{a^2 - x^2}\right] dA$ where $R = \{|x| \leq a, |y| \leq b\}$.

**12. Evaluate the following triple integrals on the given regions:

(a) $f(x,y,z) = z$ where S is the region bounded by $y^2 + z^2 = 9$ and the planes $x = 0, y = 3x$ and $z = 0$, in the first octant,

(b) $f(x,y,z) = 1$ where S is the region bounded by $y = x^2$ and the planes $z - 0, z = 4$, and $y = 9$,

(c) $f(x,y,z) = z$ where S is portion of $x^2 + y^2 + z^2 \leq 4$ in the first octant.

12.4 Change of Variables

*1. Determine the Jacobian of the following transformations. Whenever possible, write the infinitesimal area/volume element in terms of one another:

(a) $(x,y) = (e^{\xi}, \eta^3)$,

(b) $(x,y) = (5u - 2v, u + v)$,

(c) $(x,y) = (\sin(u^2v), \cos(v^2u))$,

(d) $(x,y,z) = (v + w^2, w + u^2, u + v^2)$,

(e) $(x,y,z) = (u^3 - v^2, u^3 + v^2, u^3 + v^2 + w)$.

*2. Let R be the region bounded by the curves $y = x^2$, $4y = x^2$, $xy = 1$ and $xy = 2$. Compute the integral

$$\iint\limits_R x^2y^2 dxdy.$$

*3. Determine $\displaystyle\iint\limits_S \frac{(x+y)^4}{(x-y)^5} dA$ where $S = \{-1 \leq x+y \leq 1,\ 1 \leq x-y \leq 3\}$.

*4. Compute $\displaystyle\iint\limits_R (4x{+}8y) dA$ where R is the quadrilateral with endpoints $(-1,3), (1,-3), (3,-1), (-3,1)$.

*5. Compute $\iint\limits_R \sin(9x^2+4y^2)dA$ where R is the circle $9x^2+4y^2=36$.

*6. Compute $\iint\limits_R x^2 dA$ where R is the region $a^2x^2+b^2y^2=c^2$, $a,b,c>0$.

**7. Let $R\subseteq\mathbb{R}^2$ be the region in the 1^{st} quadrant bounded by the curves $y=x^2$, $y=x^2/5$, $xy=2$, and $xy=4$.

(a) Define the variables $u=x^2/y$ and $v=xy$. Compute $dxdy$ in terms of $dudv$.

(b) Compute the area of R by changing variables from (x,y) to (u,v).

**8. (a) Let $R=[a,b]\times[c,d]$ be a rectangle and $f:R\subseteq\mathbb{R}^2\to\mathbb{R}$ be a continuous function. Assume there exist functions $f_1,f_2:\mathbb{R}\to\mathbb{R}$ such that for all $x,y\in R$, $f(x,y)=f_1(x)f_2(y)$. Show that

$$\iint\limits_R f(x,y)dA=\left[\int_{[a,b]} f_1(x)dx\right]\left[\int_{[c,d]} f_2(y)dy\right].$$

(b) It is known that the function $f(x)=e^{-x^2}$ does not have an elementary anti-derivative; however, this function is Riemann integrable on all of $\mathbb{R}$ (by using improper integrals). Compute

$$\int_{-\infty}^{\infty} e^{-x^2}dx.$$

[Hint: Consider the function $e^{-x^2-y^2}$ on $\mathbb{R}^2$ and use part (b)].

13 Vector field integration

13.1 Vector Derivatives

*1. Plot the following vector fields:

(a) $F(x,y)=(\sqrt{x},y)$,

(b) $F(x,y)=(y,x)$,

(c) $F(x,y)=\left(\frac{1}{x},y\right)$,

(d) $F(x,y)=\left(\frac{y}{x},y\right)$,

(e) $F(x,y)=(x+y,x-y)$.

*2. For each of the following functions, compute the gradient, Laplacian, divergence, and curl, if it makes sense to do so.

(a) $F(x,y)=\sin(xe^y)$,

(b) $F(x,y)=x^2+y^2-2z^2$,

(c) $F(x,y,z)=(\sin^2(x),\cos^2(z),xyz)$,

(d) $F(x,y,z)=\frac{x}{x^2+y^2}$,

(e) $F(x,y)=\log(x^2+y^2)$,

(f) $F(x,y,z)=(2xy^2z^2,2yx^2z^2,2zx^2y^2)$,

(g) $F(x,y,z,w) = (e^{zw}, e^{xz}, e^{xw}, e^{yz})$.

*3. Show that the following identities hold:

(a) $\nabla(fg) = f\nabla g + g\nabla f$

(b) $\operatorname{curl}(f\mathbf{G}) = f\operatorname{curl}\mathbf{G} + (\nabla f)\times\mathbf{G}$

(c) $\operatorname{div}(f\mathbf{G}) = f\operatorname{div}\mathbf{G} + (\nabla f)\cdot\mathbf{G}$

(d) $\operatorname{div}(\mathbf{F}\times\mathbf{G}) = \mathbf{G}\cdot(\operatorname{curl}\mathbf{F}) - \mathbf{F}\cdot(\operatorname{curl}\mathbf{G})$

*4. Show that the following identities hold:

(a) $\operatorname{curl}(\nabla f) = 0$

(b) $\operatorname{div}(\operatorname{curl}\mathbf{F}) = 0$

**5. Let (x,y) be the standard Cartesian coordinates in $\mathbb{R}^2$. Changing to polar coordinates we set

$$x = r\cos(\theta), \quad y = r\sin(\theta).$$

(a) If $e_x = (1,0)$ and $e_y = (0,1)$ are the standard Cartesian unit vectors of $\mathbb{R}^2$, determine e_r and e_θ, the standard polar unit vectors.

(b) Using the multivariable chain rule, determine the ∇ operator in polar coordinates.

(c) Compute the Laplacian ∇^2 in polar coordinates.

13.2 Line Integrals and Arc Length

*1. Find the arclength of the following curves

(a) The straight line between $(1,2,3)$ and $(3,1,2)$,

(b) The curve given by $y^2 = x^3$ between $(1,1)$ and $(4,8)$,

(c) The curve given by $(x,y) = (t - \sin(t), 1 - \cos(t))$ for $0 \le t \le 2\pi$,

(d) The curve given by $(x,y) = (\cos^3(t), \sin^3(t))$ for $0 \le t \le \frac{\pi}{2}$,

(e) The graph of a C^1 function $y = f(x)$ for $a \le x \le b$.

*2. Let $\mathbf{F}(x,y) = (-y^2, xy)$ and $C = \left\{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 : y \ge 0\right\}$. Determine $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{x}$ if C is oriented counter clockwise when viewed from $(0,0,1)$.

*3. Determine $\int_C \mathbf{F}\cdot dx$ where $\mathbf{F}(x,y) = (x^2, -y)$ and C is the graph of $y = e^x$ from $(2, e^2)$ to $(0,1)$.

*4. Determine $\int_C \mathbf{F}\cdot dx$ where $\mathbf{F}(x,y,z) = (z, -y, x)$ and C is the line segment between the points $(5,0,2)$ and $(5,3,4)$.

*5. Let $\mathbf{F}(x,y,z) = (x,y,z^2)$ and C be the curve given by the intersection of the cylinder $x^2 + y^2 = 1$ and $z = x$, with any orientation. Determine $\oint_C \mathbf{F}\cdot dx$.

*6. By explicitly parameterizing, show that if C is the constant curve (that is, the curve which consists of a single point), then for any vector field $\mathbf{F}$ we have

$$\int_C \mathbf{F}\cdot dx = 0.$$

$\star$7. Consider a vertical line segment $S = \{(x, y) : a \le y \le b, x = c\}$ in $\mathbb{R}^2$, for constants a, b, c. Show that for any vector field $\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y)$ that the line integral does not depend on F_1. Similarly conclude that for a horizontal line segment, the line integral does not depend on F_2.

$\star\star$8. Let $\mathbf{F}(x, y, z) = (yz, xz, xy)$ and define

$$C_{r,h} = \left\{(x, y, z) : x^2 + y^2 = r^2, z = h\right\}.$$

Show that for and $r > 0$ and $h \in \mathbb{R}$,

$$\int_{C_{r,h}} \mathbf{F} \cdot dx = 0.$$

13.3 Green's Theorem and Conservative Vector Fields

$\star$1. (a) Determine the values of α and β such that

$$\mathbf{F}(x, y, z) = (y + z, \alpha x + z, x + \beta y)$$

is a conservative vector field. Write down the potential function $f : \mathbb{R}^3 \to \mathbb{R}$ so that $\mathbf{F} = \nabla f$.

(b) Let C be the curve given parametrically by

$$\gamma(t) = (t\cos(t), t\sin(t), t^2), \qquad 0 \le t \le \frac{\pi}{2}$$

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$ where $\mathbf{F}$ is the your solution from part (a).

$\star$2. Compute the line integral of curve $\left\{x^2 + y^2 = 1, y \ge 0\right\}$ (oriented counter clockwise) through the vector field $\mathbf{F}(x, y) = (2xy^2 + 1, 2x^2y + 2y)$.

$\star$3. (a) Show that the vector field $\mathbf{F}(x, y, z) = (y\cos(xy), x\cos(xy) + e^z, ye^z)$ is conservative and find its scalar potential.

(b) Consider the curve C given by the intersection of the sets

$$C = \left\{x^2 + y^2 + z^2 = 4\right\} \cap \{x = 0\} \cap \{z \ge 0\}$$

oriented so that its tangent point is the positive y-direction. Determine $\int_C \mathbf{F} \cdot d\mathbf{x}$ if $\mathbf{F}$ is the vector field given in part (a).

$\star$4. Evaluate the following line integrals both directly and by using Green's Theorem.

(a) $\displaystyle\oint_C (x + 2y)dx + (x - 2y)dy$ where C is given by the union of the images of the following two functions on $[0, 1]$: $f(x) = x^2$ and $g(x) = x$, positively oriented with respect to the area the curves bound.

(b) $\displaystyle\oint_C (3x - 5y)dx + (x + 6y)dy$ where C is the ellipse $x^2/4 + y^2 = 1$ oriented counter-clockwise.

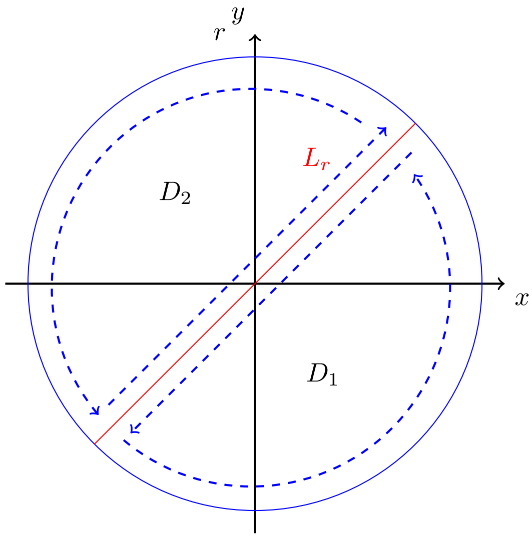
$\star\star$5. (a) Let $D \subseteq \mathbb{R}^2$ be a regular region and $\partial D = C$ be a piece-wise smooth simple closed curve, oriented positively. If $A(D)$ is the area of D, show that

$$A(D) = \oint_C x dy = \oint_C y dx = \oint_C \frac{1}{2}(y dx - x dy).$$

(b) Consider the disk of radius r, $D_r = \{x^2 + y^2 \leq r\} \subseteq \mathbb{R}^2$. Use any of the formulae from part (a) to compute the area of this disk. [Of course, you already know what the result should be!]

(c) In this question we will show that artificially adding boundaries does not affect the line integral. Let L_r be any diameter of D. Show that if we break D into two regions $D = D_1 \cup D_2$ and give the boundary of D_1 and D_2 positive orientations, then for any C^1 vector field $\mathbf{F}(x, y)$ we have

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{x} = \oint_{\partial D_1} \mathbf{F} \cdot d\mathbf{x} + \oint_{\partial D_2} \mathbf{F} \cdot d\mathbf{x}.$$



(d) Use part (a) to compute the area of the lemniscate $x^4 = x^2 - y^2$. [Be careful, as the lemniscate's boundary is not a simple closed curve.]

**6. For this problem, we will always be working in $\mathbb{R}^3$. Let $\Omega^0(\mathbb{R}^3)$ be the set of C^1 functions $\mathbb{R}^3 \to \mathbb{R}$ and $\Omega^1(\mathbb{R}^3)$ be the set of C^1 vector fields $\mathbb{R}^3 \to \mathbb{R}^3$.

(a) Show that $\Omega^0(\mathbb{R}^3)$ and $\Omega^1(\mathbb{R}^3)$ are both $\mathbb{R}$-vector spaces. [This should be short proof!]

(b) Show that grad : $\Omega^0 \to \Omega^1$ and curl : $\Omega^1 \to \Omega^1$ are linear operators.

(c) Recall that if Z is the set of closed vector fields, then $Z = \ker(\text{curl})$ and if B is the set of exact vector fields, then $B = \text{im}(\text{grad})$. Show that $B \subseteq Z$.

(d) If $S \subseteq \mathbb{R}^3$, we define the *de Rham cohomology of S of degree one* to be

$$H^1(S) = Z/B.$$

Show that if S is the complement of the z-axis in $\mathbb{R}^3$ then $H^1(S)$ is not the trivial vector space. [Hint: It suffices to show that $Z \neq B$.]

13.4 Surface Integrals and the Divergence Theorem

*1. Let S be the capless cylinder $S = \{(x, y) : x^2 + y^2 = 9, 0 \leq z \leq 5\}$, and $\mathbf{F}(x, y, z) = (2x, 2y, 2z)$. Determine the flux of $\mathbf{F}$ through S.

*2. Let S be the disk of radius 3, sitting in the plane $z = 3$. Determine the flux of $\mathbf{F}(x, y, z) = (0, 0, x^2 + y^2)$ through S.

*3. Evaluate the flux of $\mathbf{F}(x,y,z) = (3x^2, 2y, 8)$ over the plane $-2x+y+z=0$ for $(x,y) \in [0,2]\times[0,2]$, oriented pointing in the $-z$-direction.

**4. Let S be the triangle with vertices $(1,0,0), (0,2,0), (0,1,1)$, and $\mathbf{F}(x,y,z) = (xyz, xyz, 0)$. Find the flux of $\mathbf{F}$ through S.

*5. Let $S \subseteq \mathbb{R}^3$ be the capped upper half of the unit sphere; that is,

$$S = \left\{x^2+y^2+z^2 = 1, z \geq 0 \,|\, \cup\right\} \left\{x^2+y^2 \leq 1, z = 0 \,|\, .\right\}$$

Let $\mathbf{F}(x,y,z) = (2x, 2y, 2z)$ be a given vector field.

(a) Compute $\displaystyle\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dA$ using the Divergence Theorem.

(b) Compute $\displaystyle\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dA$ by parameterizing the surface explicitly. Check that this agrees with your answer from part (a).

*6. Assume that $\mathbf{F}: \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1-vector field and can be written as $\mathbf{F} = \operatorname{curl}\mathbf{G}$ for some C^2-vector field G. Show that if S is any piecwise smooth surface which bounds a regular region in $\mathbb{R}^3$, then the flux of $\mathbf{F}$ through S is zero.

*7. Let $\mathbf{F}(x,y,z) = (xy, yz, xz)$ and $S = \left\{x^2+y^2 \leq 1, 0 \leq z \leq 1\right\}$ be the solid cylinder.

(a) Directly (without using any theorems) compute the flux of $\mathbf{F}$ through ∂S if ∂S has the Stokes' orientation relative to S.

(b) Compute the flux of $\mathbf{F}$ through S by using the Divergence theorem.

**8. (a) Let $E: \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$E(x,y,z) = \frac{k}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$$

for some $k > 0$. Show that $\operatorname{div} E = 0$.

(b) Let E be the vector-field given in part (b), and let $S_r = \{x^2+y^2+z^2 = r^2\} \subseteq \mathbb{R}^3$ be the sphere of radius r. Compute the flux of E through S_r.

(c) You got different answers in part (b) and (c). Explain why this is not a contradiction to the Divergence Theorem.

**9. Consider the cube

$$C = \{0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq z \leq c \,|\, ,\} \qquad a, b, c > 0.$$

and let $\mathbf{F}(x,y,z) = [(x-a)yz, x(y-b)z, xy(z-c)]$.

(a) Directly compute the flux of $\mathbf{F}$ through ∂C; that is, compute $\iint_{\partial C} \mathbf{F}\cdot\hat{\mathbf{n}}\ dA$. [Hint: Use symmetry to reduce this whole computation to a single integral.]

(b) Directly compute the integral $\iiint_C \operatorname{div}\mathbf{F} dV$.

(c) These two quantities are equal by a theorem. State that theorem.

***10. Let $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x}))$ be a vector field in $\mathbb{R}^3$.

(a) For arbitrary $h > 0$, let $S_h = \left\{(x, y, z) : x^2 + y^2 + z^2 = h^2\right\}$ be the sphere of radius h. Parameterize S_h by a function $\mathbf{g} : [a, b] \times [c, d] \to \mathbb{R}^3$. Compute $\frac{\partial \mathbf{g}}{\partial s} \times \frac{\partial \mathbf{g}}{\partial t}$.

(b) Under the assumption that h is very small, we can use a first order approximation on the functions F_i. Write out the linear approximations for $F_i(\mathbf{x})$ at $(0, 0, 0)$ and evaluate these on the parameterization.

(c) Use parts (a) and (b) to determine $\mathbf{F}(\mathbf{g}(t)) \cdot \left(\frac{\partial \mathbf{g}}{\partial s} \times \frac{\partial \mathbf{g}}{\partial t}\right)$. [Ignore terms in order h^4, or keep track of them by writing $O(h^4)$]

(d) Compute $\displaystyle\lim_{h\to 0} \frac{1}{\frac{4}{3}\pi h^3} \oiint_{S_h} \mathbf{F} \cdot \hat{\mathbf{n}} dS$. Compare this to the divergence. Conclude that divergence is the infinitesimal flux. [Notice that $\frac{4}{3} pih^3$ is the volume of the sphere, so we are 'normalizing' by the volume in our limit.]

13.5 Stokes Theorem

*1. Let C be the circle $\left\{(x, y, 0) : x^2 + y^2 = r^2\right\}$ for some $r > 0$, and set $\mathbf{F}(x, y, z) = (x, x, y)$. Determine $\oint_C \mathbf{F} \cdot d\mathbf{x}$.

*2. Let C be the curve which is the intersection of the cylinder $\left\{(x, y, z) : x^2 + y^2 = 1\right\}$ with the plane $z = 0$, in the first octant; that is, C is a quarter circle in the xy-plane. Let $\mathbf{F}(x, y, z) = (y, z, x)$.

(a) By explicitly parameterizing the curve, determine $\oint_C \mathbf{F} \cdot d\mathbf{x}$,

(b) Evaluate the line integral using Stokes theorem and confirm that you get the same answer.

*3. Let $\mathbf{F}(x, y, z) = (z-y, -x-z, -x-y)$ and C be the curve given by the intersection of $x^2+y^2+z^2 = 4$ and the plane $z = y$; oriented counter clockwise when viewed from $(0, 0, 1)$.

(a) By explicit computation, determine $\oint_C \mathbf{F} \cdot d\mathbf{x}$,

(b) Using Stokes theorem, verify your result from part (a).

**4. Let S be a smooth oriented surface in $\mathbb{R}^3$ with piecewise smooth, compatibly oriented boundary ∂S. Show that if f in C^1 and g is C^2 on S then

$$\int_{\partial S} f \nabla g \cdot d\mathbf{x} = \iint_S (\nabla f \times \nabla g) \cdot \hat{\mathbf{n}}\, dA.$$

***5. (a) Let S_1 and S_2 be smooth surfaces in $\mathbb{R}^3$ with piecewise smooth boundary satisfying $\partial S_1 = \partial S_2$. If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field, and S_1 and S_2 are oriented so that the Stokes orientation on ∂S_1 is the same as that of ∂S_2, show that

$$\iint_{S_1} (\operatorname{curl} \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA = \iint_{S_2} (\operatorname{curl} \mathbf{F}) \cdot \hat{\mathbf{n}}\, dA.$$

(b) Assume that $S \subseteq \mathbb{R}^4$ is an oriented smooth surface with boundary

$$\partial S = \{(x, y, 0) : x^2 + y^2 = 1\}.$$

If $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field such that $(\operatorname{curl} \mathbf{F}) \cdot (0, 0, 1) = 0$, show that

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \hat{\mathbf{n}} \, dA = 0.$$

**6. Show that if $S \subseteq \mathbb{R}^3$ is a smooth, closed, boundaryless surface oriented so that the unit normal points outwards, and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1-vector field, then

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \hat{\mathbf{n}} \, dA = 0.$$

[Hint: Draw any closed curve C in S and use this to break S into two disjoint parts. Now think about what the Stokes orientation is on C]

Note: If you cannot do the general case, try the following special case: Let $R \subseteq \mathbb{R}^3$ be such that $\partial R = S$ and assume that $\mathbf{F}$ is C^1 on R.

***7. Let $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x}))$ be a vector field in $\mathbb{R}^3$.

(a) For arbitrary $h > 0$, let $S_h = \left\{(x, y, 0) : x^2 + y^2 = h^2\right\}$ be the circle of radius h in the xy-plane. Parameterize S_h by a function $\mathbf{g} : [a, b] \to \mathbb{R}^3$.

(b) Under the assumption that h is very small, we can use a first order approximation on the functions F_i. Write out the linear approximations for $F_i(\mathbf{x})$ at $(0, 0, 0)$ and evaluate these on the parameterization.

(c) Use parts (a) and (b) to determine $\mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t)$. [Ignore terms in order h^3, or keep track of them by writing $O(h^3)$]

(d) Compute $\displaystyle\lim_{h\to 0} \frac{1}{\pi h^2} \oint_{S_h} \mathbf{F} \cdot d\mathbf{x}$. Compare this to the curl. Conclude that curl is the infinitesimal circulation. [Notice that πh^2 is the area of the circle, so we are 'normalizing' by the area in our limit.]

13.6 Differential Forms

*1. Determine $d\omega$ where ω is given

(a) $\omega = e^x dx + zy dy + x^2 \cos(y) dz$ in $\mathbb{R}^3$,

(b) $\omega = (x + z) dx \wedge dy + y^2 z^2 dx \wedge dz + \cos(zx) dy \wedge dx$,

(c) $\omega = xzw dx + y \cos(w) dy$ in $\mathbb{R}^4$.

**2. (a) Show that $d^2 = 0$ by explicitly computing $d^2 f$ for a 0-form $f : S \subseteq \mathbb{R}^3 \to \mathbb{R}$.

(b) Show that $d^2 = 0$ by explicitly computing $d^2\omega$ for ω a 1-form in $\mathbb{R}^3$.

*3. Compute the following pullbacks $f^*\omega$:

(a) Let $\omega = xy dx + 2z dy - y dz$ and $(x, y, z) = f(u, v) = (uv, u^2, 3u + v)$,

(b) Let $\omega = e^{xy} dx + w dy + x^2 dz + dw$ and $(x, y, z, w) = f(t, u, v) = ((v + t)u, t, u, v)$.

**4. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) = x^2 + y^2 + z^2$ and let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$F(u, v) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} \right).$$

Compute $F^* df$ and $d(f \circ F)$ separately and verify that they are equal.

***5. Let $f : \mathbb{R}^k \to \mathbb{R}$ be a 0-form and $F : \mathbb{R}^n \to \mathbb{R}^k$. Show that $F^* df = d(f \circ F)$.

**6. A k-form ω is said to be decomposible if $\omega = \omega_1 \wedge \cdots \wedge \omega_k$ where the ω_k are one-forms. The form is said to be indecomposible otherwise.

(a) Show that $dx \wedge dy + dx \wedge dz + dy \wedge dz$ is decomposible in $\mathbb{R}^3$,

(b) Show that $dx \wedge dy + dz \wedge dw$ is indecomposible.