University of Toronto

Faculty of Arts and Science

Quiz 4

MAT2371Y - Advanced Calculus

Duration - 50 minutes

No Aids Permitted

Surname: ______________________________

First Name: ______________________________

Student Number: ______________________________

Tutorial:

T0101	**T5101**	**T5102**
T4/R4	**T5/R5**	**T5/R5**
SS1074	**SS1070**	**BA1240**

This exam contains 4 pages (including this cover page) and 3 problems. Check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include, but are not limited to textbooks, notes, calculators, or any electronic device.

Unless otherwise indicated, you are required to show your work on each problem on this exam. The following rules apply:

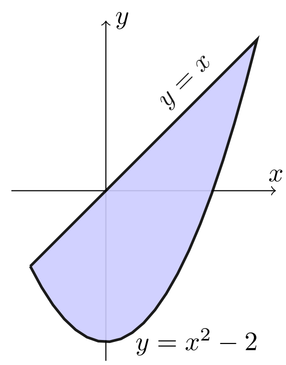
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	10	
2	10	
3	10	
Total:	30	

1. (10 points) Determine $\iint_S xy dA$ where S is the region

$$S = \left\{(x, y) : y \geq x^2 - 2 \text{ and } y < x\right\}.$$

Solution: We first deduce where the two curves meet. Setting $x^2 - 1 = x$ we get $x^2 - x - 2 = (x-2)(x+1) = 0$, giving $x = 2$ and $x = -1$. This corresponds to the two points $(-1, -1)$ and $(2, 2)$. We plot the region below:



We can write this as an x-simple set

$$S = \left\{(x, y) : -1 \leq x \leq 2,\, x^2 - 2 \leq y \leq x\right\}.$$

Setting up our iterated integral and solving, we get

$$\begin{aligned}
\iint_S xy dA &= \int_{-1}^{2} \int_{x^2-2}^{x} xy\, dy dx = \int_{-1}^{2} \left[\frac{xy^2}{2}\right]_{y=x^2-2}^{y=x} dx \\
&= \int_{-1}^{2} \frac{x}{2} \left[x^2 - (x^4 - 4x^2 + 4)\right] dx \\
&= \frac{1}{2} \int_{-1}^{2} \left(-x^5 + 3x^3 - 4x\right) dx = \frac{9}{8}
\end{aligned}$$

2. (10 points) Let R be the region bounded by the curves $y = x^2$, $4y = x^2$, $xy = 1$ and $xy = 2$. Compute the integral

$$\iint_R x^2 y^2 dxdy.$$

Solution: Consider the map $(u, v) = g(x, y) = (x^2/y, xy)$ which takes $R \to [1, 4] \times [1, 2]$. The Jacobian (scaling factor) is then

$$\begin{aligned}
|\det Dg| &= \left| \det \begin{pmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ y & x \end{pmatrix} \right| \\
&= \left| \frac{2x^2}{y} + \frac{x^2}{y} \right| \\
&= 3\frac{x^2}{y}.
\end{aligned}$$

Notice that we do not need the absolute values since $y > 0$ on R. Hence this tells us that $du\ dv = 3\frac{x^2}{y} dx\ dy$, but we were asked to find $dx\ dy$ in terms of $du\ dv$. Recognizing that under the change of variable, $3\frac{x^2}{y} = 3u$ we can just divide to get

$$dx\ dy = \frac{1}{3u} du\ dv.$$

Our integrand x^2y^2 is just v^2, so under change of variable we get

$$\begin{aligned}
\iint_R x^2 y^2 dA &= \int_1^4 \int_1^2 \frac{v^2}{3u} dudv \\
&= \frac{1}{3} \left[\int_1^4 \frac{1}{u} du \right] \left[\int_1^2 v^2 dv \right] \\
&= \frac{7}{9} \log |4|
\end{aligned}$$

3. (10 points) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 functions. Show that $\operatorname{div}(f\mathbf{G}) = f \operatorname{div}(\mathbf{G}) + \nabla f \cdot \mathbf{G}$.

Solution: Let $\mathbf{G}(x, y) = (G_1(x, y), G_2(x, y))$, so that $(f\mathbf{G})(x, y) = (f(x, y)G_1(x, y), f(x, y)G_2(x, y))$. Applying the divergence operator we get:

$$\begin{aligned}
\operatorname{div}(f\mathbf{G}) &= \left[\frac{\partial}{x} f(x,y)G_1(x,y)\right] + \left[\frac{\partial}{y} f(x,y)G_2(x,y)\right] \\
&= \partial_x f G_1 + f\partial_x G_1 + \partial_y f G_2 + f\partial_y G_3 \\
&= f\,\underbrace{(\partial_x G_1 + \partial_y G_2)}_{\operatorname{div}(\mathbf{G})} + (\partial_x f, \partial_y f) \cdot (G_1, G_2) \\
&= f \operatorname{div}(\mathbf{G}) + \nabla f \cdot \mathbf{G}
\end{aligned}$$

which is what we wanted to show.