# MAT237 - Tutorial 5 - 2 June 2015

## 1 Coverage

**Compactness**: They know everything from this section. We're just going to be doing one problem from this.

**Connectedness**: They've now learned everything from this section of the notes, including path connectedness.

**Parametrized Subsets of $\mathbb{R}^n$**: They've covered up to the example in the lecture notes about parametrizing the unit circle.

## 2 Problems

I suggest the following problems.

1. (BL 5.4) Given two sets $U, V \subseteq \mathbb{R}^n$, define the *distance* between them by

$$d(U, V) = \inf \{ \|x - y\| \,:\, x \in U, y \in V \}.$$

   (a) Show that if either $\overline{U} \cap V \neq \emptyset$ or $U \cap \overline{V} = \emptyset$, then $d(U, V) = 0$.

   (b) Show that if $U$ is compact, $V$ is closed, and $U \cap V = \emptyset$, then $d(U, V) > 0$.

   (c) Show that the compactness of $U$ was necessary in the previous part by giving an example of two disjoint, closed subsets of $\mathbb{R}^2$ with zero distance between them.

2. (BL 6.3, expanded a bit) Let's explore how different sorts of connected sets interact.

   (a) Let $A$ and $B$ be connected subsets of $\mathbb{R}^n$. Must $A \cup B$ and $A \cap B$ be connected? Prove or give counterexamples.

   (b) Let $A$ and $B$ be path connected subsets of $\mathbb{R}^n$. Must $A \cup B$ and $A \cap B$ be path connected, or even just connected? Prove or give counterexamples.

   (c) Let $A$ and $B$ be convex subsets of $\mathbb{R}^n$. Must $A \cup B$ and $A \cap B$ be convex, or even just path connected, or *even* just connected? Prove or give counterexamples.

3. Is $\mathbb{R} \setminus \mathbb{Q}$ connected? Is $\mathbb{R}^n \setminus \mathbb{Q}^n$ connected, for $n \geq 2$?

4. (BL 7.6) Let $C$ be the curve at the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + z = 1$. Give a parametrization of $C$.

## 3 Solutions and Comments

1. ***Solution***: (a) Assume that $\overline{U} \cap V \neq \emptyset$, and fix an $x$ in this intersection. Then since $x \in \overline{U}$, there is a sequence $\{x_k\} \subseteq U$ converging to $x$. Then for any $n > 0$, we can in particular find a $k$ such that $\|x_k - x\| < \frac{1}{n}$. Since $x_k \in U$ and $x \in V$, this shows that $d(U, V) < \frac{1}{n}$. Since we can do this for any $n$, it must be the case that $d(U, V) = 0$. The case where $U \cap \overline{V} \neq \emptyset$ is analogous.

   (b) We prove this by contrapositive. Assume that $U$ is compact, $U \cap V = \emptyset$, and $d(U, V) = 0$. We show that $V$ must not be closed.

   Since $d(U, V) = 0$, there must exist sequence $\{x_k\} \subseteq U$ and $\{y_k\} \subseteq V$ such that $\|x_k - y_k\| \to 0$. Since $U$ is compact, there is a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ converging to some point $x$. We show that the corresponding subsequence $\{y_{k_n}\}$ of $\{y_k\}$ converges to the same point $x$. This shows that $V$ is not closed, since a sequence from $V$ is converging to a point outside $V$. To see this, note:

$$\|y_{k_n} - x\| = \|y_{k_n} - x_{k_n} + x_{k_n} - x\| \leq \|y_{k_n} - x_{k_n}\| + \|x_{k_n} - x\| \to 0$$

   An alternate, nicer proof is as follows:

   We first show that the function $f : U \to \mathbb{R}$ given by $f(x) = d(x, V) = \inf \{ \|x - y\| \, : \, y \in V \}$ is continuous. Indeed, given $x_1, x_2 \in U$ and $y \in V$, by the triangle equality we have:

$$\|x_1 - y\| \leq \|x_2 - y\| + \|x_1 - x_2\|.$$

   Taking infima we get:

$$d(x_1, V) \leq d(x_2, V) + \|x_1 - x_2\|.$$

   This rearranges to show $|d(x_1, V) - d(x_2, V)| \leq \|x_1 - x_2\|$, from which it follows that the function is continuous (it's 1-Lipschitz).

   Now, if $U$ is compact, then since the function defined above is continuous, by the extreme value theorem $f$ achieves its minimum at some point $x_0 \in U$. Since $x_0 \notin V$ and $V$ is closed, $d(x_0, V) = d(U, V) > 0$.

   (c) The easiest example is something like the two components of the graph of $f(x) = \frac{1}{x^2}$ in $\mathbb{R}^2$.

   ***Comments***: I chose to put this problem on here because people seemed to be having real trouble with it on Piazza. Even with the first part, which I would have thought would be straightforward. My hope, at least, is that people have some intuitive idea of the argument in the first part, and just can't write it down. I think people are getting intimidated by the infimum and that's clouding their judgment.

   Either of the proofs in part (b) are likely more than they can be expected to come up with. Certainly the second one is. (I included that one just because it's nice.) The proof in the first one isn't *so* hard to imagine, at least. The students probably won't know where to start with the question. They won't know whether to prove it by contradiction or contrapositive, or even how a proof like that would go. Once you decide to prove it the way I did, by assuming $U$ is compact and showing $V$ is not closed, the rest of it completely writes itself. Again, I wouldn't stress this part too much. It's pretty hard.

Part (c) is a great question! If people come up with an answer like this, they'll be very pleased with themselves. Examples like the one I gave (all examples that I know are of this flavour) really give you a good intuitive feeling for why the boundedness part of compactness is so important. This question should probably come before (b), now that I think about it...

2. ***Solution***: First of all, all of these problems have trivial cases where the two sets are disjoint, and then things are vacuously as connected as you want. Note that, but in what follows I always assume $U \cap V \neq \emptyset$.

   (a) $A \cup B$ must be connected: Let $S_1$ and $S_2$ be a potential disconnection of $A \cup B$. Since $A$ is connected, either $A \subseteq S_1$ or $A \subseteq S_2$. Without loss, say it's the first one. Since $B$ is connected, we must also have $B \subseteq S_1$ or $B \subseteq S_2$. It can't be a subset of $S_2$, since $S_1 \cap S_2 = \emptyset$, and so we must have $S_2 = \emptyset$.

   $A \cap B$ need not be connected. One imagines two V-shaped sets opening towards one another. They're both connected, but their intersection has two widely separated pieces.

   (b) $A \cup B$ must be path connected: Join any point from $A$ to a point in $A \cap B$, and in turn to any point in $B$.

   $A \cap B$ need not be path connected or even connected. The example in the previous case is a good one.

   (c) $A \cup B$ need not be convex. For example, take two non-parallel lines in $\mathbb{R}^2$. It must be path connected by the previous part though.

   $A \cap B$ must be convex: Take two points in the intersection. The straight line joining them lies in both sets since they're convex, and so lies in their intersection.

   ***Comments***: With this, we just want them to get their hands dirty with connectedness a bit. The way it's defined for them is weird and technical looking, while we know that connectedness is maybe the most intuitive topological property a set can have. The only thing that's at all tricky here is the proof that the union of two connected sets is connected, and even that is intuitively obvious.

   Some things to stress here are how different $\mathbb{R}$ and $\mathbb{R}^n$ for $n \geq 2$ behave with respect to these questions. They have a theorem that says the only connected subsets of $\mathbb{R}$ are intervals, so connected and convex are the same there. It would also be a good idea to see if they can come up with subsets of $\mathbb{R}^2$ that are path connected but not convex (which is easy) or connected but not path connected (which is tricky). In higher dimensions sets have room to wiggle around which they don't have in $\mathbb{R}$. That "room to wiggle" is a recurring theme that underlies why all this calculus business is so much trickier in higher dimensions.

3. ***Solution***: The irrationals are very much not connected. $(\mathbb{R} \setminus \mathbb{Q}) \cap (-\infty, q)$ and $(\mathbb{R} \setminus \mathbb{Q}) \cap (q, \infty)$ forms a disconnection for any $q \in \mathbb{Q}$, as is easy to check.

On the other hand, $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected. Given two points in there, you can go from one to the other along axis-parallel lines.

***Comments***: This is another quick question showing how different connectedness looks in higher dimensions. The set of irrationals in $\mathbb{R}$ is totally disconnected (in the technical sense, meaning the only connected components are singletons), while the "corresponding" subset of the plane is path connected.

4. ***Solution***: First let's find the set of points comprising $C$. To do this, substitute the equation of the plane into the equation of the sphere. This yields:

$$\begin{aligned} x^2 + y^2 + (1-x)^2 &= 1 \\ x^2 + y^2 + 1 - 2x + x^2 &= 1 \\ 2x^2 - 2x + y^2 &= 0 \end{aligned}$$

Completing the square, this is: $2(x-\frac{1}{2})^2 + y^2 = \frac{1}{2}$. This is the equation of a circle of radius $\frac{1}{4}$ in the coordinates $u = \sqrt{2}(x - \frac{1}{2})$, $v = y$, so we can parametrize it by

$$\begin{cases} \sqrt{2}(x - \frac{1}{2}) = \frac{1}{4}\cos t \\ y = \frac{1}{4}\sin t \end{cases}$$

for $t \in [0, 2\pi]$. We then solve this for $x$ and $y$:

$$\begin{cases} x = \frac{1}{4\sqrt{2}}\cos t + \frac{1}{2} \\ y = \frac{1}{4}\sin t \end{cases}$$

This gives us a parametrization of the $x$ and $y$ coordinates, so all that remains is to plug back into the equation of the plane to get the $z$ coordinate:

$$z = 1 - x = 1 - \left(\frac{1}{4\sqrt{2}}\cos t + \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{4\sqrt{2}}\cos t.$$

Our final parametrization is therefore:

$$(x(t), y(t), z(t)) = \left(\frac{1}{4\sqrt{2}}\cos t + \frac{1}{2}, \frac{1}{4}\sin t, \frac{1}{2} - \frac{1}{4\sqrt{2}}\cos t\right) \text{ for } t \in [0, 2\pi].$$

***Comments***: Finally, a computational problem for a tutorial! I guess I don't have much to say about it. They're new to parametrizations at this point, so go through this slowly and deliberately. It gets more complicated when we're parametrizing higher-dimensional objects, so it would be nice if they really get this one. Build them up before we inevitably knock them down.