MAT237 - Tutorial 9 - 16 June 2015

1 Coverage

Taylor polynomials, multi-index notation.

2 Problems

I suggest the following problems.

1. (BL 9.4 (iii)) Find the third order Taylor polynomial of $f(x, y) = \log(1 + x - y)$ about the origin.

2. (BL 9.3) Prove the following version of the binomial theorem: Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. If α is any multi-index, then:

$$(x+y)^\alpha = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} x^\beta y^\gamma,$$

   where β and γ are also multi-indices, and the addition is component-wise. (Hint: Use induction on n. The base case is the binomial theorem.)

3. With any remaining time (or if they really want to know), I suggest talking to them about parametrizations and tangent spaces some more. The students in my tutorial are strong, and still aren't totally comfortable with these. The examples in BL 7.2 that have hyperbolic cross sections are scary for them. I didn't get to cover this last tutorial, so I may bring it up again this time.

3 Solutions and Comments

1. ***Solution***: Given a point $h = (x, y)$ near the origin, we need to calculate all the terms in the sum:

$$\sum_{|\alpha|\leq 3} \frac{\partial^\alpha f(0,0)}{\alpha!} h^\alpha.$$

   We'll make a table like in the lecture notes.

$\|\alpha\|$	α	$\partial^\alpha f$	$\alpha!$	$h^c\alpha$	$\frac{\partial^\alpha f(0,0)}{\alpha!}$
0	$(0,0)$	$\log(1+x-y)$	$0!0!=1$	1	0
1	$(1,0)$	$\partial_x f = \frac{1}{1+x-y}$	$1!0!=1$	x	x
1	$(0,1)$	$\partial_y f = -\frac{1}{1+x-y}$	$0!1!=1$	y	$-y$
2	$(2,0)$	$\partial_x^2 f = -\frac{1}{(1+x-y)^2}$	$2!0!=2$	x^2	$-\frac{x^2}{2}$
2	$(1,1)$	$\partial_x\partial_y f = \frac{1}{(1+x-y)^2}$	$1!1!=1$	xy	xy
2	$(0,2)$	$\partial_y^2 f = -\frac{1}{(1+x-y)^2}$	$0!2!=2$	y^2	$-\frac{y^2}{2}$
3	$(3,0)$	$\partial_x^3 f = \frac{2}{(1+x-y)^3}$	$3!0!=6$	x^3	$\frac{x^3}{3}$
3	$(2,1)$	$\partial_x^2\partial_y f = -\frac{2}{(1+x-y)^3}$	$2!1!=2$	x^2y	$-x^2y$
3	$(1,2)$	$\partial_x\partial_y^2 f = \frac{2}{(1+x-y)^3}$	$1!2!=2$	xy^2	xy^2
3	$(0,3)$	$\partial_y^3 f = -\frac{2}{(1+x-y)^3}$	$0!3!=6$	y^3	$-\frac{y^3}{3}$

So, our Taylor polynomial is the sum of the terms in the last column:

$$P_{(0,0),3}(x,y) = x - y - \frac{x^2}{2} + xy - \frac{y^2}{2} + \frac{x^3}{3} - x^2y + xy^2 - \frac{y^3}{3}.$$

Notice the pleasing fact that:

$$P_{(0,0),3}(x,y) = (x-y) - \frac{1}{2}(x-y)^2 + \frac{1}{3}(x-y)^3$$

which is exactly the 3rd order Taylor polynomial for $\log(1+x)$ with $x-y$ substituted into it.

Comments: A straightforward computation exercise. The hope is to get them familiar with multi-index notation, and show them that these Taylor polynomials aren't scary at all. This is probably the most straightforward thing they've learned so far.

2. ***Solution***: This proposition is proved by repeated application of the usual binomial theorem in an induction on n. As the hint says, the base case is precisely the usual binomial theorem.

   So, assume the result for $n-1 \geq 1$, and we'll prove it for n. Fix a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then by definition,

$$(x+y)^\alpha = (x_n+y_n)^{\alpha_n}(x'+y')^{\alpha'},$$

   where $x' = (x_1, \ldots, x_{n-1}, 0)$ and similarly for y', and $\alpha' = (\alpha_1, \ldots, \alpha_{n-1}, 0)$. By the inductive hypothesis, we know that

$$(x'+y')^{\alpha'} = \sum_{\beta+\gamma=\alpha'} \frac{(\alpha')!}{\beta!\gamma!}(x')^\beta(y')^\gamma,$$

and by the usual binomial theorem we know that

$$(x_n + y_n)^{\alpha_n} = \sum_{k=0}^{\alpha_n} \frac{(\alpha_n)!}{k!(\alpha_n - k)!}(x_n)^k (y_n)^{\alpha_n - k}.$$

Now, a given pair of multi-indices β and γ satisfying $\beta + \gamma = \alpha$ must be of the form $\beta = (\beta_1, \dots, \beta_{n-1}, k)$ and $\gamma = (\gamma_1, \dots, \gamma_{n-1}, \alpha_n - k)$ for some $k = 0, \dots, \alpha_n$. But then the multi-indices $\beta' = (\beta_1, \dots, \beta_{n-1}, 0)$ and $\gamma' = (\gamma_1, \dots, \gamma_{n-1}, 0)$ appear in the sum above, and when we multiply the two sums, all possible products between terms in each sum appear.

In particular, the following product will appear:

$$\left(\frac{(\alpha_n)!}{k!(\alpha_n - k)!}(x_n)^k (y_n)^{\alpha_n - k}\right)\left(\frac{(\alpha')!}{(\beta')!(\gamma')!}(x')^{\beta'}(y')^{\gamma'}\right) = \frac{\alpha!}{\beta!\gamma!}x^\beta y^\gamma.$$

Comments: This problem is easy, but it's an enormous mess of notation. My solution above is worded so that another mathematician would believe it, but for the students you probably want to explain a little more why what I did shows the result, explaining why all the terms you want to appear in this giant product of sums do in fact appear, and why there are no extras. This is purely an exercise in understanding multi-index notation.

Be sure to remind them of the usual binomial theorem. They should all have heard of it, but they might know it only with binomial coefficients rather than written out in factorials like that. It's perhaps worth telling them that the sum in the question can be rewritten with binomial coefficients:

$$(x+y)^\alpha = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!}x^\beta\, y^\gamma = \sum_{\beta+\gamma=\alpha} \binom{\alpha_1}{\beta_1} \cdots \binom{\alpha_n}{\beta_n} x^\beta y^\gamma$$

It's also worth noting that the previous problem on the Big List is almost exactly the same problem, but about a generalization of the product rule rather than a generalization of the binomial theorem. They're the same argument though, exactly. I chose to do this one because most of the students are probably familiar with the binomial theorem, while not all of them will have thought about the generalized product rule for single variable functions.