MAT237 - Tutorial 18 - 30 July 2015

1 Coverage

Surface integrals, but not the divergence theorem.

2 Problems

I suggest the following problems. Like last time, I won't have too much to comment about any of these, since they're mostly computational.

I'm only suggesting two problems because frankly there isn't much to talk about. Knowing surface integrals but not the divergence theorem means there's nothing to do other than compute tedious surface integrals (like the two below).

I suggest using whatever time is left over to answer any lingering questions they may have or help with problems from the previous section they haven't quite gotten. Another idea might be to derive the two-dimensional Divergence Theorem, which is equivalent and very similar to Green's Theorem, and is nice by way of preface to the full Divergence Theorem which they'll learn in class today.

1. (BL 13.4.4) Let S be the triangle with vertices $(1,0,0)$, $(0,2,0)$, $(0,1,1)$, and let $\mathbf{F}(x,y,z) = (xyz, xyz, 0)$. Find the flux of $\mathbf{F}$ through S.

2. (BL 13.4.5 (b)) Let S be the capped upper half unit sphere with the Stokes orientation. That is
$$\{\,(x,y,z) \,:\, x^2+y^2+z^2 = 1,\ z \geq 0\,\} \cup \{\,(x,y,z) \,:\, x^2+y^2 \leq 1,\ z = 0\,\},$$
and let $\mathbf{F}(x,y,z) = (2x, 2y, 2z)$. Compute $\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dA$ by explicitly parametrizing S.

3. From Green's Theorem, derive the two-dimensional version of the Divergence Theorem. That is, given a region $D \subseteq \mathbb{R}^2$ with a piecewise smooth simple closed boundary, and a vector field $\mathbf{F}$ defined on an open set containing D, then
$$\oint_{\partial D} \mathbf{F}\cdot\hat{\mathbf{n}}\,ds = \int\int_D \nabla\cdot\mathbf{F}\,dA.$$

3 Solutions and Comments

1. ***Solution***: First we should parametrize the surface. This surface is a portion of a plane, so we first write the equation of the plane. Two vectors on this plane are $(1,0,0)-(0,2,0) = (1,-2,0)$ and $(1,0,0)-(0,1,1) = (1,-1,-1)$. Then the vector $(1,-2,0)\times(1,-1,-1) = (2,1,1)$ is normal to the plane, and so this plane has equation $2x+y+z-2=0$.

We can therefore parameterize our triangle as

$$G(x,y) = (x, y, 2-2x-y) \quad \text{where} \quad 0 \le x \le 1, \;\; 1-x \le y \le 2-2x.$$

We compute:

$$\frac{\partial G}{\partial x} = (1, 0, -2), \quad \frac{\partial G}{\partial y} = (0, 1, -1),$$

and as expected this yields $\frac{\partial G}{\partial x} \times \frac{\partial G}{\partial y} = (2, 1, 1)$.

Next, substituting this into our function, we get:

$$\begin{aligned}
\mathbf{F}(G(x,y)) \cdot \left(\frac{\partial G}{\partial x} \times \frac{\partial G}{\partial y} \right) &= \mathbf{F}(G(x,y)) \cdot (2,1,1) \\
&= (2xy - 2x^2y - xy^2, 2xy - 2x^2y - xy^2, 0) \cdot (2,1,1) \\
&= 3(2xy - 2x^2y - xy^2)
\end{aligned}$$

Finally, we can integrate:

$$\int\int_S \mathbf{F} \cdot \hat{\mathbf{n}} \, dA = \int_0^1 \int_{1-x}^{2-2x} 3(2xy - 2x^2y - xy^2) \, dy dx$$

This integral is tedious but elementary to compute. It ends up equaling $\frac{1}{10}$.

It's possible that this is less messy with a smarter parameterization, but I chose the easiest one to write down.

2. ***Solution***: Call the spherical part S_1 and the disc part S_2. we'll parameterize and integrate each of them separately. First, S_1.

   The simplest way to parameterize the sphere is using spherical coordinates. That is, $G(\theta, \phi) = (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))$, where $0 \le \theta \le 2\pi$, and $0 \le \phi \le \frac{\pi}{2}$. Then we have:

$$\frac{\partial G}{\partial \theta} = (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0), \quad \frac{\partial G}{\partial \phi} = (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi)),$$

   from which we compute:

$$\frac{\partial G}{\partial \theta} \times \frac{\partial G}{\partial \phi} = -(\cos(\theta)\sin^2(\phi), \sin(\theta)\sin^2(\phi), \sin(\phi)\cos(\phi)).$$

   Note that these vectors point towards the inside of the sphere, meaning they are oriented opposite to what we need. We'll fix this by omitting the initial minus sign in the integral below.

So, the integral for S_1 is:

$$
\begin{aligned}
&\int\int_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,dA \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 2(\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))\cdot(\cos(\theta)\sin^2(\phi), \sin(\theta)\sin^2(\phi), \sin(\phi)\cos(\phi))\,d\phi d\theta \\
&= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos^2(\theta)\sin^3(\phi) + \sin^2(\theta)\sin^3(\phi) + \sin(\phi)\cos^2(\phi)\,d\phi d\theta \\
&= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin^3(\phi) + \sin(\phi)\cos^2(\phi)\,d\phi d\theta \\
&= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin(\phi)\,d\phi d\theta \\
&= 4\pi.
\end{aligned}
$$

Second, we do the calculation for the simpler surface S_2. This is readily parameterized by $G(r,\theta) = (r\cos(\theta), r\sin(\theta), 0)$ in the usual way. This yields:

$$\frac{\partial G}{\partial r} = (\cos(\theta), \sin(\theta), 0), \quad \frac{\partial G}{\partial \theta} = (-r\sin(\theta), r\cos(\theta), 0),$$

from which we compute

$$\frac{\partial G}{\partial r}\times\frac{\partial G}{\partial \theta} = (0,0,r).$$

This normal vector points upwards, which again the opposite of what we need, so we instead will use $(0,0,-r)$ in the calculation below.

$$\int\int_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,dA = \int_0^{2\pi}\int_0^1 2(r\cos(\theta), r\sin(\theta), 0)\cdot(0,0,-r)\,dr d\theta = 0.$$

So, our final answer is $\int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dA = 4\pi$.

The students don't know the Divergence Theorem yet, but you can hint at it here because of how tremendously much easier this problem is with it. The divergence of $\mathbf{F}$ is just 6, and so the Divergence Theorem tells us that the surface integral in this problem is equal to 6 times the volume of the half sphere, which is $6\cdot\frac{2}{3}\pi = 4\pi$.

3. ***Solution***: Fix such a region D, and let $\gamma(t) = (x(t), y(t))$ parameterize ∂D on $[0,1]$. We know from before that $\gamma'(t) = (x'(t), y'(t))$ gives us a tangent vector to the curve at every point.

   In this problem however, we want an outward normal vector to the curve at every point. That can easily be obtained by switching the coordinates and adding a minus sign to the second coordinate. That is, $(y'(t), -x'(t))$ is an outward normal vector to our curve. Normalizing, we have our unit normal vector:

$$\hat{\mathbf{n}} = \frac{1}{\|\gamma'(t)\|}(y'(t), -x'(t)).$$

Now we can evaluate the left side of the expression in the theorem:

$$\begin{aligned}
\oint_{\partial D} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds &= \int_0^1 \mathbf{F}(\gamma(t)) \cdot \frac{1}{\|\gamma'(t)\|}(y'(t), -x'(t))\, \|\gamma'(t)\|\, dt \\
&= \int_0^1 \mathbf{F}(\gamma(t)) \cdot (y'(t), -x'(t))\, dt \\
&= \oint_{\partial D} -F_2\, dx + F_1\, dy,
\end{aligned}$$

where F_1 and F_2 are the components of $\mathbf{F}$, as usual. This line integral can also be seen as $\oint_{\partial D} \mathbf{G} \cdot d\mathbf{x}$ where $\mathbf{G} = (-F_2, F_1)$.

Now apply Green's Theorem to this line integral to obtain:

$$\oint_{\partial D} -F_2\, dx + F_1\, dy = \int\int_D \frac{\partial F_1}{\partial x} - \frac{\partial(-F_2)}{\partial y}\, dA = \int\int_D \nabla \cdot \mathbf{F}\, dA,$$

as required.