Tutorial 2 - MAT244 - C.J. Adkins

Chapter 2

Methods for solving 1st Order O.D.E

① Seperable Equations

$$M(x)=N(y)\frac{dy}{dx} \quad \text{where } M,N:\mathbb{R}\to\mathbb{R}$$

This form is nice since we can "split" the O.D.E into an integral Equation:

$$M(x)=N(y)\frac{dy}{dx} \iff \int M(x)dx=\int N(y)dy$$

Ex (2.2-#8) Solve

$$\frac{dy}{dx}=\frac{x^2}{1+y^2}$$

Use method of Seperation so

$(C\in\mathbb{R})$

$$y'=\frac{x^2}{1+y^2} \iff y'(1+y^2)=x^2 \iff \int(1+y^2)dy=\int x^2dx \iff y+\frac{y^3}{3}=\frac{x^3}{3}+C$$

Remark: Sometimes it's impossible to get y just in terms of x (i.e what is y(x)?) So we leave it as an implicit solution.

Ex (2.2-#20) Solve:

$$y^2(1-x^2)^{1/2}dy=\arcsin x\, dx$$

Note this is in differential form, this is equivalent to

$$y^2(1-x^2)^{1/2}y'=\arcsin x$$

To solve it we seperate the Eq.

let $u=\arcsin x$, $du=\frac{dx}{\sqrt{1-x^2}}$

$$y^2(1-x^2)^{1/2}y'=\arcsin x \iff \int y^2dy=\int\frac{\arcsin x}{\sqrt{1-x^2}}dx \iff \frac{y^3}{3}=\int u\,du=\frac{u^2}{2}+C=\frac{\arcsin^2 x}{2}+C$$

$$\therefore y(x)=\sqrt[3]{\frac{\arcsin^2 x}{2}+C}$$

② Integrating Factor

$$y'+p(x)y=q(x)$$

If the O.D.E has the above form (Any 1st Order Linear O.D.E) then consider

$$\frac{d}{dx}(I(x)y(x))=y'I+I'y \quad \text{where } I(x)=\exp\left(\int p(x)dx\right)$$

$$=I(x)\left(y'+p(x)y\right)$$

This $I(x)$ factor allows us to simplify the O.D.E to

$$y' + p(x)y = q(x) \iff \frac{d}{dx}(I y) = I(x)q(x)$$

If we integrate, we obtain

$$y(x) = \frac{1}{I(x)}\int I(x)q(x)\,dx$$

This is a nice closed-form solution for $y(x)$

Ex (2.1-#20) Solve the I.V.P $ty' + (t+1)y = t$ s.t $y(\ln(2)) = 1$, $t > 0$

Put it in normal form: $y' + \underbrace{\frac{(t+1)}{t}}_{p(t)}y = \underbrace{1}_{q(t)}$, our $I(t) = \exp\left[\int p(t)\,dt\right] = \exp(t + \ln t) = te^t$

Using the formula from before we see that:

$$y(t) = \frac{1}{te^t}\underbrace{\int te^t\,dt}_{\text{By parts w/ } u=t,\ dv=e^t dt,\ du=dt,\ v=e^t} = \frac{1}{te^t}\left(e^t(t-1) + C\right) = 1 - \frac{1}{t} + \frac{C}{te^t}$$

The initial conditions say $y(\ln 2) = 1$, this gives us

$$y(\ln 2) = 1 - \frac{1}{\ln(2)} + \frac{C}{2\ln(2)} = 1 \iff C = 2$$

$$\therefore y(t) = 1 - \frac{1}{t} + \frac{2}{te^t}$$

Ex (2.1-#38-Variation of Parameters)

a) if $y' + p(t)y = q(t)$ and $q(t) = 0\ \forall t$, What is $y(t)$? By Seperation of Variables we have

$$y' = -p(t)y \iff \int \frac{dy}{y} = -\int p(t)\,dt \iff \ln(y) = -\int p(t)\,dt \iff y(t) = C\exp\left(-\int p(t)\,dt\right)$$

b) if q is not everywhere zero, then allow the constant to vary w/ t i.e

$$y \longrightarrow C(t)\exp\left(-\int p(t)\,dt\right) = C(t)\tilde{I}(t) = C(t)\frac{1}{I(t)}$$

What is the condition on $C(t)$?

To find this, plug into the O.D.E

$$y' + py = q \longrightarrow C'\tilde{I} + \underbrace{C\tilde{I}'}_{-Cp\tilde{I}} + pC\tilde{I} = C'\tilde{I} = q \iff C'(t) = qI(t) \iff C(t) = \int q(t)I(t)\,dt$$

Notice that this gives the solution we found earlier $y(t) = \frac{1}{I(t)}\int q(t)I(t)\,dt$

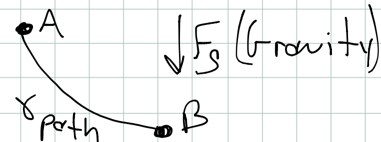
Ex (2.1 #40) Solve: $y' + \frac{1}{t}y = 3\cos(2t)$

We use our integrating factor method: $I(x) = \exp(\int p(x)dx) = x$

Therefore the formula gives: $y(t) = \frac{1}{t}\underbrace{\int 3t\cos(2t)\,dt}_{\text{By parts twice.}} = \frac{1}{t}\frac{3}{4}\left(2t\sin(2t) + \cos(2t) + C\right)$

Modeling w/ 1^{st} order (Word Problems)

Ex(2.3-#32-Brachistochrone)



Question: What is the best path γ to minimize the time of $A \to B$ under gravity with no friction.

Remark: To find the O.D.E that models this we'd need "The Calculus of Variations"!!!

i) Solve the O.D.E $(1+y'^2)y = k^2$ (this is what you'd get via variations)

$(1+y'^2)y = k^2 \iff y'^2 = \frac{k^2}{y} - 1 \Rightarrow y' = \sqrt{\frac{k^2}{y} - 1}$ why do we take + root (A: Want $A \to B$, not $B \to A$)

b) Introduce $y = k^2\sin^2 t$, see what happens.

$y' = 2k^2 \sin t \cos t\, t'$, plug in $y' = \sqrt{\frac{k^2}{y} - 1} \longrightarrow 2k^2\sin t\cos t = \sqrt{\frac{k^2}{k^2\sin^2 t} - 1} = \sqrt{\frac{1-\sin^2 t}{\sin^2 t}} = \frac{\cos t}{\sin t} \iff 2k^2\sin^2 t\, t' = 1$

c) Change variables again to $\theta = 2t$ & solve

$\therefore 2k^2\sin^2 t\, t' = 1 \iff k^2 \sin^2\left(\frac{\theta}{2}\right)\theta' = 1 \iff k^2\int \underbrace{\sin^2\left(\frac{\theta}{2}\right)}_{\text{halfangle } \frac{1}{2} - \frac{\cos\theta}{2}} d\theta = \int dx \iff x = k^2(\theta - \sin\theta)/2 + C$

If we fix A at $(0,0)$ & reverse dependence (i.e $y \leftrightarrow x$) we can show y's derivation

Together, we see $x = k^2\frac{(\theta - \sin\theta)}{2}$ & $y = k^2\frac{(1-\cos\theta)}{2}$

This graph is called a cycloid. Now if we pick $B = (x_0, y_0)$ we can find k

Questions & Quiz Time