Tutorial 5 - MAT244 - C.J. Adkins

2nd Order Equations

2nd Order linear O.D.E take the form

$$y'' + py' + qy = \begin{cases} 0 & \leftarrow \text{Homogeneous} \\ g(t) & \leftarrow \text{NonHomogeneous} \end{cases}$$ (this is just the math lingo)

How to solve? It depends on many things, so we'll go case by case.

1) Homogeneous w/ Constant Coefficients

$$ay'' + by' + cy = 0 \quad \text{where } a,b,c \in \mathbb{R}$$

Consider the solution $y = \exp(\lambda t)$, if we plug this in, we obtain

$$\exp(\lambda t)\left(a\lambda^2 + b\lambda + c\right) = 0$$

The exponential function is never zero, so this will work if

$$a\lambda^2 + b\lambda + c = 0 \quad \text{(this is called the characteristic equation)}$$

Remember the roots of a quadratic are given by

$$\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This produces 3 types of solutions depending on $b^2 - 4ac = \Delta$ (discriminant)

① If $\Delta > 0 \Rightarrow$ the roots are real, so $y(t) = A\exp(\lambda_+ t) + B\exp(\lambda_- t)$, where $A, B \in \mathbb{R}$

② If $\Delta < 0 \Rightarrow$ the roots are complex, thus by Euler's formula:

$$y(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t} = \exp\left(\frac{-b}{2a}t\right)\left(A\exp\left(i\frac{\sqrt{4ac-b^2}}{2a}t\right) + B\exp\left(-i\frac{\sqrt{4ac-b^2}}{2a}t\right)\right)$$

$$= \exp\left(\frac{-b}{2a}t\right)\left(\tilde{A}\sin\left(\frac{\sqrt{4ac-b^2}}{2a}t\right) + \tilde{B}\cos\left(\frac{\sqrt{4ac-b^2}}{2a}t\right)\right)$$

We'll deal with the 3rd case later on.

Ex (3.1-#15) Solve: $y'' + 8y' - 9y = 0$, $y(1) = 1$, $y'(1) = 0$

By the above formula, $\Delta > 0$, so $\lambda_\pm = \frac{-4 \pm \sqrt{100}}{2} = -4 \pm 5$

$$\Rightarrow y(t) = Ae^{t} + Be^{-9t}$$

Now we find A & B w/ the initial data.

$y(1) = 1 \Rightarrow 1 = Ae + Be^{-9}$, $y'(1) = 0 \Rightarrow 0 = Ae - 9Be^{-9} \Rightarrow 1 = 10Be^{-9} \Leftrightarrow B = \frac{e^9}{10}$

$\Rightarrow \quad 1 = Ae + \frac{1}{10} \Rightarrow \frac{9}{10e} = A \qquad \therefore \boxed{y(t) = \frac{1}{10}\left(9e^{t-1} + e^{9-9t}\right)}$

Basically the formula does all the work!

Ex (3.3-#22) Solve: $y'' + 2y' + 2y = 0,\ y(\frac{\pi}{4}) = 2,\ y'(\frac{\pi}{4}) = -2$

By the above formula, $\Delta < 0$, so

$$y(t) = \exp(-t)\left(\tilde{A}\sin(t) + \tilde{B}\cos(t)\right)$$

$y(\frac{\pi}{4}) = 2 \Rightarrow \exp(-\frac{\pi}{4})\left(\frac{\tilde{A}+\tilde{B}}{\sqrt{2}}\right) = 2,\ y'(\frac{\pi}{4}) = -2 \Rightarrow -\exp(-\frac{\pi}{4})\left(\frac{\tilde{A}+\tilde{B}}{\sqrt{2}}\right) + \exp(-\frac{\pi}{4})\left(\frac{\tilde{A}-\tilde{B}}{\sqrt{2}}\right) = -2$

$\Rightarrow \tilde{B} = \sqrt{2}\exp(\pi/4)$

$\Rightarrow \tilde{A} = \sqrt{2}\exp(\pi/4)$

$$\therefore y(t) = \sqrt{2}\exp\left(\frac{\pi}{4} - t\right)\left(\sin(t) + \cos(t)\right)$$

Ex (3.3-#34) (Euler Equations) Lets try to solve:

$$t^2 y'' + \alpha t y' + \beta y = 0 \text{ where } \alpha, \beta \in \mathbb{R},\ t > 0 \text{ (this is called an Euler Eq)}$$

a) change variables by $t \to \ln(t) = x$, we need to calculate y'', y', in terms of x by chain rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\frac{1}{t},\quad \frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}\frac{1}{t}\right) = \frac{d^2y}{dx^2}\frac{1}{t^2} - \frac{dy}{dx}\frac{1}{t^2}$$

If we plug this in we obtain

$$t^2\left(\frac{1}{t^2}\frac{d^2y}{dx^2} - \frac{1}{t^2}\frac{dy}{dx}\right) + \alpha t\left(\frac{1}{t}\frac{dy}{dx}\right) + \beta y = 0 \iff \frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0$$

This has constant coefficients so $y(x) = y(\ln(t)) \Rightarrow y(t) = At^{\lambda_+} + Bt^{\lambda_-}$ where $\lambda_\pm = \frac{1-\alpha \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$

Ex (3.3-#40) Solve: $t^2 y'' + 7ty' + 10y = 0,\ t > 0$

By the above we try $y = t^\lambda$, this implies $t^\lambda(\lambda(\lambda-1) + 7\lambda + 10) = 0$, but $t^\lambda \neq 0$, so....

$\lambda^2 + 6\lambda + 10 = 0 \Rightarrow \lambda_\pm = -3 \pm \sqrt{9-10} = -3 \pm i \Rightarrow y(t) = \frac{1}{t^3}(At^i + Bt^{-i}) = \frac{1}{t^3}(\tilde{A}\sin(\ln(t)) + \tilde{B}\cos(\ln(t)))$

Now for theory: Theorem: If we have $y'' + py' + qy = g(t)$ w/ $y(t_0) = y_0$, $y'(t_0) = y_0'$ and $p, g, \& q$ are continuous in some interval I that contains t_0. Then there's one solution to the equation above in I.

No proof, but it basically means these equations have answers (may not be pretty or even analytic)

Ex: (3.2-#9) Where is the largest interval where the solution exists? $t(t-4)y'' + 3ty' + 4y = 2,\ y(3) = 0,\ y'(3) = -1$

write in s.f $\Rightarrow y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$, we need $3 \in I$, and there are blow ups at $t=0$ & $t=4$, so

$I = (0, 4)$

Wronskian: A tool for linear independence. It is defined as $W(y_1,y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1y_2' - y_1'y_2$

Notice it is a function of t if y is a function of t.

Theorem: y_1 & y_2 are linearly independent iff $W(y_1,y_2)(t) \neq 0$ $\forall t$ defined.

Thus, if y_1 & y_2 solve a 2nd order linear equation & $W(y_1,y_2) \neq 0 \Rightarrow y = Ay_1 + By_2$ is the general solution.

Theorem: (Abel's) If y_1, y_2 solve $y'' + py' + qy = 0$ w/ p & q continuous on "I" then

$$W(y_1,y_2)(t) = A\exp\left[-\int p(t)\,dt\right]$$

Proof: $W = y_1y_2' - y_1'y_2 \Rightarrow W' = y_1y_2'' - y_2y_1''$, if we plug in the O.D.E, we see

$$W' + pW = 0 \Rightarrow W = A\exp\left[-\int p\,dt\right]$$ ∎

Ex (3.2-#28) Consider $y'' - y' - 2y = 0$

a) By the characteristic Eq, we know $y_1 = e^{-t}$ & $y_2 = e^{2t}$.

Notice that $W(y_1,y_2) = y_1y_2' - y_1'y_2 = 2e^{t} + e^{t} = 3e^{t} \neq 0 \Rightarrow y_1$ & y_2 are fundamental!

b) If $y_3 = -2e^{2t}$, $y_4 = y_1 + 2y_2$, $y_5 = 2y_1 - 2y_3$

Are these solutions to the O.D.E? Yes, by linearity (super position)

c) Are the following sets fundamental?

$[y_1, y_3]$? Yes, $W(y_1,y_3) \neq 0$, $[y_2,y_3]$? No, since $W(y_2,y_3) = 0$

$[y_1,y_4]$? Yes, $W(y_1,y_4) \neq 0$ $\quad$ $[y_4,y_5]$? No, since:

$y_5 = 2y_1 + 4y_2$, $y_4 = y_1 + 2y_2 \Rightarrow y_5 = 2y_4 \Rightarrow W(y_4,y_5) = 0$

Ex (3.2-#41) (Exact 2nd Order)

$$P(x)y'' + Q(x)y' + R(x)y = 0 \text{ exact} \Leftrightarrow [P(x)y']' + [f(x)y]' = 0$$

Can integrate to first order: $P(x)y' + f(x)y = \text{const}$

Notice that $[Py']' + [fy]' = P'y' + Py'' + f'y + fy' = P(x)y'' + (P'+f)y' + f'y = 0$

$\Rightarrow f' = R \Rightarrow f = \int R(x)\,dx$, what about Q? $Q = P' + \int R(x)$ This implies we need

$$P'' - Q' + R = 0 \text{ for exactness.}$$ ∎

Quiz Question: What is $W(y_1,y_2)$ for y_1,y_2 s.t. they solve $(1+t)y'' - y' + ty = 0$

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