MAT244 - Ordinary Differential Equations - Summer 2016
Assignment 2   Due: July 20, 2016

Full Name: ______________________________________

Last   First

Student #: ______________________________________

Indicate which Tutorial Section you attend by filling in the appropriate circle:

◯ Tut 01 M 2-3 pm BA1240 Christopher Adkins

◯ Tut 02 M 3-4 pm BA1240 Fang Shalev Housfater

◯ Tut 05 W 5-6 pm BA2165 Krishan Rajaratnam

Print out this page, fill it out and attach it to the front of your assignment.

Instructions:

- Due July 20, 2016 at the start of the lecture 13:10pm in BA1170.
- You may collaborate with your classmates but you MUST write up your solutions independently.
- Write your solutions clearly, showing all steps. Do not submit just your rough work. Grading is based on both the correctness and thee presentation of your answer.
- Late assignments will not be accepted without appropriate documentation to explain the lateness (eg. a UofT medical note)
- Assignments may be submitted to the course instructor for remarking up to one week after they are returned. If you request a regrade, please attach a note explaining clearly which part and why you believe it was graded incorrectly.

For grader use:

Q1	Q2	Q3	Q4	Q5	Total

each problem is worth 20 points for a total of 100.

1) Find the general solution to

$$y'' - 4y' + 4y = e^{2x} \arctan(2x)$$

Solution: Since the nonhomogeneity is not of the form of a quasipolynomial, we must use the method of variation of parameters (that works for any nonhomogeneity). To do this, we first compute the solution of the homogeneous equation

$$y'' - 4y' + 4y = 0$$

Plugging in $y = e^{\lambda x}$ gives an equation for λ:

$$\lambda^2 - 4\lambda + 4 = 0$$

or by factoring, $(\lambda - 2)^2 = 0$. Thus $\lambda = 2$ is the only eigenvalue and we have by the usual trick the general solution to the homogenous problem:

$$y_h = c_1 e^{2x} + c_2 x e^{2x}$$

with c_1, c_2 given by initial conditions. From here, we denote the independent solutions to the homogeneous problem as

$$\begin{aligned} y_h^{(1)} &= e^{2x} \\ y_h^{(2)} &= x e^{2x} \end{aligned}$$

Computing the Wronskian of these solutions, we get

$$\begin{aligned} W &= \det \begin{pmatrix} y_h^{(1)} & y_h^{(2)} \\ y_h^{(1)\prime} & y_h^{(2)\prime} \end{pmatrix} \\ &= \det \begin{pmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & (1+2x) e^{2x} \end{pmatrix} \\ &= e^{4x} (1 + 2x - 2x) \\ &= e^{4x} \end{aligned}$$

By variation of parameters, the particular solution is given by

$$\begin{aligned} y^{(p)} &= -y_h^{(1)} \int \frac{y_h^{(2)} g}{W} + y_h^{(2)} \int \frac{y_h^{(1)} g}{W} \\ &= -e^{2x} \int \frac{x e^{2x} e^{2x} \arctan(2x)}{e^{4x}} + x e^{2x} \int \frac{e^{2x} e^{2x} \arctan(2x)}{e^{4x}} \end{aligned} \tag{1}$$

The second integral seems easier so let's start with that one and integrate by parts:

$$\begin{aligned}\int \frac{e^{2x}e^{2x}\arctan(2x)}{e^{4x}} &= \int \arctan(2x)\\ &= x\arctan(2x) - \int x\frac{2}{1+(2x)^2}\\ &= x\arctan(2x) - \frac{1}{4}\int\frac{8x}{1+4x^2}\\ &= x\arctan(2x) - \frac{1}{4}\ln\left(1+4x^2\right)\end{aligned}$$

We use this result in the first integral in (1) to get again by intee

$$\begin{aligned}\int \frac{xe^{2x}e^{2x}\arctan(2x)}{e^{4x}} &= \int x\arctan(2x)\\ &= \frac{x^2}{2}\arctan(2x) - \int\frac{x^2}{1+4x^2}\\ &= \frac{x^2}{2}\arctan(2x) - \frac{1}{4}\int\frac{4x^2+1}{1+4x^2} + \frac{1}{4}\int\frac{1}{1+4x^2}\\ &= \frac{x^2}{2}\arctan(2x) - \frac{x}{4} + \frac{1}{8}\arctan(2x)\end{aligned}$$

Combining these two integrations, we have

$$y^{(p)} = -e^{2x}\left(-\frac{x}{4} + \frac{1}{8}\arctan(2x) + \frac{x^2}{2}\arctan(2x)\right) + xe^{2x}\left(x\arctan(2x) - \frac{1}{4}\ln\left(1+4x^2\right)\right)$$

The general solution is composed by adding a homogeneous part with constants depending on the initial conditions:

$$\begin{aligned}y(x) &= c_1e^{2x} + c_2xe^{2x}\\ &\quad - e^{2x}\left(-\frac{x}{4} + \frac{1}{8}\arctan(2x) + \frac{x^2}{2}\arctan(2x)\right) + xe^{2x}\left(x\arctan(2x) - \frac{1}{4}\ln\left(1+4x^2\right)\right)\end{aligned}$$

2) Consider the equation

$$4(x+1)^2\frac{d^2y}{dx^2} + 10(x+1)\frac{dy}{dx} + \frac{27}{16}y = (x+1)^3 \tag{2}$$

a) Show that for $x < -1$, (2) this is equivalent to

$$4\frac{d^2y}{dt^2}+6\frac{dy}{dt}+\frac{27}{16}y=-e^{3t} \tag{3}$$

where $t=\ln|x+1|$

Solution: Using $t=\ln|x+1|$ as suggested, we note that for $x<-1$, $t=\ln(-x-1)$. We now carefully change variables in equation (2) (using the chain rule!):

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{dt}\frac{dt}{dx}\\&=\frac{dy}{dt}\frac{1}{x+1}\end{aligned}$$

To compute the second derivative involves applying the product rule so let's be extra careful:

$$\begin{aligned}\frac{d^2y}{dx^2}&=\frac{d}{dx}\left(\frac{dy}{dx}\right)\\&=\frac{d}{dx}\left(\frac{1}{x+1}\frac{dy}{dt}\right)\\&=-\frac{1}{(x+1)^2}\frac{dy}{dt}+\frac{1}{x+1}\frac{d}{dx}\frac{dy}{dt}\\&=-\frac{1}{(x+1)^2}\frac{dy}{dt}+\frac{1}{x+1}\frac{d^2y}{dt^2}\frac{1}{x+1}\end{aligned}$$

the last line follows again from the chain rule. Putting all this information together, we obtain the result as predicted (note the negative on the $\frac{dy}{dt}$ term):

$$4\frac{d^2y}{dt^2}+6\frac{dy}{dt}+\frac{27}{16}y=(x+1)^3$$

the only thing left to do is to change the nonhomegeneity to a function of t to be consistent throughout. Since $(x+1)=-(-x-1)$ we have that $(x+1)^3=-e^{3t}$ (just following the rules of exponents.)

Finally we get the equation of constant coefficents that we are asked for:

$$4\frac{d^2y}{dt^2}+6\frac{dy}{dt}+\frac{27}{16}y=-e^{3t}$$

b) Find the general solution $y(t)$ of (3) (Which method is easier? you decide)

To solve the constant coefficient equation (3), we look for a solution of the form

$$y(t)=e^{\lambda t}$$

and get the characteristic equation:

$$4\lambda^2 + 6\lambda + \frac{27}{16} = 0$$

Using the quadratic formula to solve for λ gives

$$\begin{aligned}\lambda &= \frac{1}{8}\left(-6 \pm \sqrt{6^2 - 4 * 4 * \frac{27}{16}}\right) \\ &= \frac{1}{8}\left(-6 \pm \sqrt{36 - 37}\right) \\ &= \frac{1}{8}\left(-6 \pm 3\right)\end{aligned}$$

So that independent solutions $y(t)$ to the homogeneous problem are

$$y_h = c_1 e^{-\frac{3}{8}t} + c_2 e^{-\frac{9}{8}t}$$

Now the nonhomogeneous part $-e^{3t}$ does not appear in the homogeneous solution, so we look for a particular solution of(3) in the form $y^{(p)} = Ae^{3t}$. Plugging this into (3) gives

$$\left(4 * 3^2 A + 6 * 3A + \frac{27}{16}A\right) e^{3t} = -e^{3t}$$

So what if the coefficient on the left hand side is not nice, we call

$$36 + 18 + \frac{27}{16} = k \tag{4}$$

and the particular solution solves

$$\begin{aligned}kA &= -1 \\ A &= \frac{-1}{k}\end{aligned}$$

Therefore the general solution to (3) is

$$y = c_1 e^{-\frac{3}{8}t} + c_2 e^{-\frac{9}{8}t} - \frac{1}{k}e^{3t}$$

c) Find the solution $y(x)$ of (2) corresponding to $y(x = -2) = y'(x = -2) = 0$ (Hint: This is much simpler if you use part b) rather than variation of parameters directly on (2). But you can convince yourself that both give the same answer!)

Solution:

As per the hint, we translate the general solution $y(t)$ to the solution of the original equation (2) using the change of variables $t = \ln(-x-1)$. Note that when $x = -2$, $t = \ln(-x-1) = \ln(2-1) = \ln 1 = 0$. Therefore the solution in terms of t becomes

$$\begin{cases} y(t=0) & = c_1 + c_2 - \frac{1}{k} = 0 \\ y'(t=0) & = -\frac{3}{8}c_1 - \frac{9}{8}c_2 - \frac{3}{k} = 0 \end{cases}$$

We can solve this system to obtain c_1 and c_2 to get

$$c_1 = \frac{1}{k} - c_2$$

$$c_2 = -\frac{27}{6}k$$

$$c_1 = \frac{33}{6}k$$

(Note if you have a complicated constant like $k = 36 + 18 + \frac{27}{16}$ there is no shame in hiding it to avoid hideous calculations)

The solution to $y(t)$ thus becomes

$$y(t) = \frac{33}{6}ke^{-\frac{3}{8}t} - \frac{27}{6}ke^{-\frac{9}{8}t} - \frac{1}{k}e^{3t}$$

Recalling the definition $t = \ln(-x-1)$, we have the solution of the nonhomogeneous euler equation as

$y(x) = \frac{33}{6}k\,|x+1|^{-3/8} - \frac{27}{6}k\,|x+1|^{-9/8} + \frac{1}{k}(x+1)^3$

(I do not do the variation of parameters directly on euler's equation because it is too much work. Exercise: check that it is the same solution)

3) Draw an accurate phase portrait for the following systems of equations. Justify your portrait (by computing eigenvalues and vectors! If it is a spiral, which direction will it spin?)

a) $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$

Solution: We find the eigenvalues of A by solving the characteristic equation

$$
\begin{aligned}
0 &= \det(A - \lambda I) \\
&= \det \begin{pmatrix} -\lambda & -3 \\ 3 & -\lambda \end{pmatrix} \\
&= \lambda^2 + 3
\end{aligned}
$$

Thus $\lambda^2 = -9$ and the eigenvalues are purely imaginary: $\lambda = \pm i3$. To solve for the eigenvector, we set

$$
\begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 3i \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}
$$

In other words, we have

$$
\begin{cases} -3\xi_2 & = 3i\xi_1 \\ 3\xi_1 & = 3i\xi_2 \end{cases}
$$

Both of these equations are the same and reduce to

$$
\xi_1 = i\xi_2
$$

So if we choose $\xi_2 = 1$, then $\xi_1 = i$ and one of our solutions reads

$$
\xi^{(1)} e^{3it} = \begin{pmatrix} i \\ 1 \end{pmatrix} e^{3it}
$$

and the other independent solution is the complex conjugate:

$$
\xi^{(2)} e^{-3it} = \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{-3it} \tag{5}
$$

By Problem 5 of the midterm we can add and subtract independent solutions to get another pair of indy solutions. For example, we write

$$
\begin{aligned}
\xi^{(1)} e^{3it} &= \begin{pmatrix} i \\ 1 \end{pmatrix} e^{3it} \\
&= \begin{pmatrix} i \\ 1 \end{pmatrix} (\cos(3t) + i\sin(3t)) \\
&= \begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix} + i \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix}
\end{aligned}
$$

and the other independent solution is the complex conjugate. We can thus add and subtract to obtain independent solutions in terms of only sin and cos (functions I understand unlike ie^{3it} and such!).
Thus the general solution can be written as

$$
\mathbf{y} = c_1 \begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix} + c_2 \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} \tag{6}
$$

The phase portrait is a center that spins counter clockwise (since $a_{21} = 3 > 0$) See Figure 1 for an approximate phase portrait

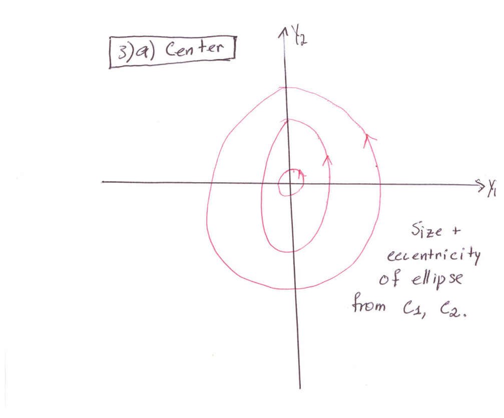

Figure 1: An approximate phase portrait for the system in 3)a)

b) $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$

Solution:

We find the eigenvalues of A, solving

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{pmatrix}1-\lambda & 2\\ -2 & 1-\lambda\end{pmatrix}\\ &= (1-\lambda)^2+4\\ &= \lambda^2-2\lambda+5=0\end{aligned}$$

Using the quadratic formula,

$$\begin{aligned}\lambda &= \frac{1}{2}\left(2\pm\sqrt{4-20}\right)\\ &= 1\pm\frac{1}{2}\sqrt{-16}\\ &= 1\pm 2i\end{aligned}$$

Finding the eigenvector $\xi^{(1)}$ for $\lambda = 1+2i$, we find:

$$\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = (1+2i)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

$$\begin{cases}\xi_1+2\xi_2 & = (1+2i)\,\xi_1\\ -2\xi_1+\xi_2 & = (1+2i)\,\xi_2\end{cases} \tag{7}$$

The first equation in (7) simplifies to

$$\xi_2 = i\xi_1$$

and the second to

$$-\xi_1 = i\xi_2$$

These equations being a multiple of each other. We can therefore choose $\xi^{(1)} = \begin{pmatrix}1\\ i\end{pmatrix}$ with $\xi^{(2)}$ the complex conjugate. One independent solution to this equation is thus

$$y_1 = \begin{pmatrix}1\\ i\end{pmatrix}e^{(1+2i)t}$$

Separate y_1 into its real and imaginary components to get:

$$
\begin{aligned}
y_1 &= e^t \begin{pmatrix} 1 \\ i \end{pmatrix} e^{2it} \\
&= e^t \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos(2t) + i\sin(2t)) \\
&= e^t \left[\begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + i \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} \right]
\end{aligned}
$$

Therefore the general solution can be written only in terms of exponentials and trigonometric functions:

$$
y = e^t \left[c_1 \begin{pmatrix} \cos(2t) \\ -\sin(2t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2t) \\ \cos(2t) \end{pmatrix} \right]
$$

with c_1, c_2 determined by initial conditions. You can recognize this solution as an unstable spiral spinning away from zero. Since the off-diagonal element of A is $a_{21} = -2$, the spiral spins clockwise. See figure 2 for a typical trajectory

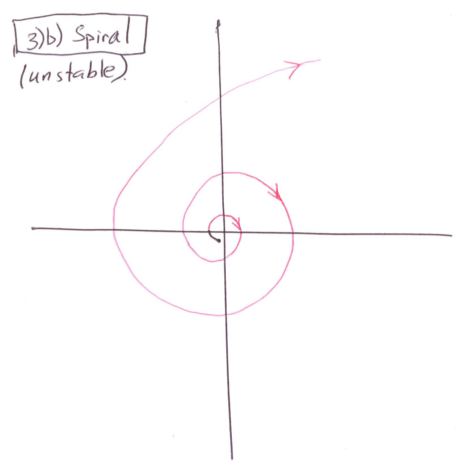

Figure 2: An approximate phase portrait for the system in 3)b)

c) $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{pmatrix} -6 & -5 \\ 5 & 4 \end{pmatrix}$

Solution: Finding the eigenvalues of A as usual, we get

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{pmatrix}-6-\lambda & -5\\ 5 & 4-\lambda\end{pmatrix}\\ &= (-6-\lambda)(4-\lambda)+25\\ &= \lambda^2+2\lambda+1\\ &= (\lambda+1)^2 = 0\end{aligned}$$

Thus $\lambda = -1$ is the only eigenvalue. We find the corresponding eigenvector ξ:

$$\begin{pmatrix}-6 & -5\\ 5 & 4\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = -\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$$

$$\begin{cases}-6\xi_1 - 5\xi_2 & = -\xi_1\\ 5\xi_1 + 4\xi_2 & = -\xi_2\end{cases} \tag{8}$$

Both equations in (8) reduce to

$$\xi_1 + \xi_2 = 0$$

and we can choose the eigenvector to be $\xi = \begin{pmatrix}1\\ -1\end{pmatrix}$. Thus one solution to this equation is just

$$y_1 = \begin{pmatrix}1\\ -1\end{pmatrix} e^{-t}$$

To find another independent solution, we write

$$y_2 = \xi t e^{-t} + \eta e^{-t}$$

Where $\xi = \begin{pmatrix}1\\ -1\end{pmatrix}$ is our eigenvector and η is a generalized eigenvector solving $(A-\lambda I)\eta = \xi$ In our case this is:

$$\begin{pmatrix}-5 & -5\\ 5 & 5\end{pmatrix}\eta = \begin{pmatrix}1\\ -1\end{pmatrix}$$

Thus we have $5(\eta_1+\eta_2) = -1$. We have the freedom to set $\eta_2 = 0$ and write $\eta = \begin{pmatrix}-1/5\\ 0\end{pmatrix}$. The general solution is therefore

$$y = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{-t} + \begin{pmatrix} -1/5 \\ 0 \end{pmatrix} e^{-t} \right)$$

See 3 for an approximate phase portrait

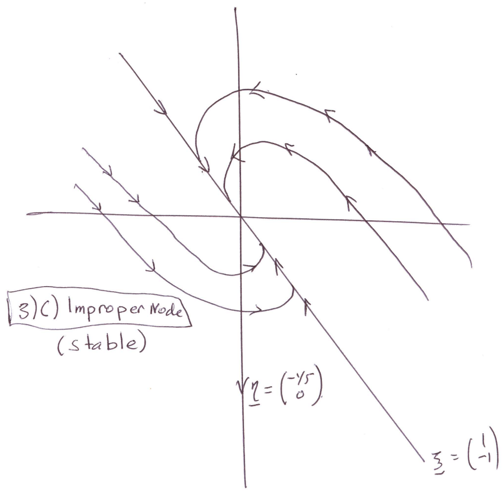

Figure 3: An approximate phase portrait for the system in 3)c)

4) a) Find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

We solve for the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} \\ &= (1-\lambda)(-2-\lambda) - 4 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda+3)(\lambda-2) \end{aligned}$$

Thus the eigenvalues are $\lambda_1 = -3$, and $\lambda_2 = 2$. We find first $\xi^{(1)}$ corresponding to the eigenvalue λ_1:

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = -3 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\begin{cases} \xi_1 + \xi_2 &= -3\xi_1 \\ 4\xi_1 - 2\xi_2 &= -3\xi_2 \end{cases} \tag{9}$$

Both of the equations in (9) reduce to $4\xi_1 + \xi_2 = 0$. Choosing $\xi_1 = 1$, we may set

$$\xi^{(\mathbf{1})} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

Similarly, we find the eigenvector $\xi^{(\mathbf{2})}$ corresponding to λ_2:

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 2 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

$$\begin{cases} \xi_1 + \xi_2 & = 2\xi_1 \\ 4\xi_1 - 2\xi_2 & = 2\xi_2 \end{cases} \tag{10}$$

Both of the equations in (10) reduce to $\xi_1 - \xi_2 = 0$ so we can choose:

$$\xi^{(\mathbf{2})} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b) Use these to compute the special fundamental matrix $e^{\mathbf{At}}$. (note this matrix is sometimes denoted $\mathbf{\Phi(t)}$ in section 7.7. See our derivation in the notes).

Solution: We learned 3 ways to find the special fundamental matrix e^{At}. I will take the second approach but all of them have lead to the same result. By part a), the general solution to $\mathbf{y}' = Ay$ can be written as

$$y = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

for some c_1, c_2 determined by initial conditions. Similarly, the matrix e^{At} also solves the same equation:

$$\frac{d}{dt} e^{At} = Ae^{At}$$

with the initial condition $e^{At}\big|_{t=0} = I$ where I is the identity matrix. Our problem then reduces to finding two solutions corresponding to $y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $y_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We have for the first choice:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solving for c_1, c_2 gives

$$c_1 = 1/5, \quad c_2 = 4/5$$

Similarly, we solve for the constants such that

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solving for c_1, c_2 in this case gives

$$c_1 = -1/5, \quad c_2 = 1/5$$

Thus the special fundamental matrix is

$$\begin{aligned} e^{At} &= \left(1/5 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + 4/5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}, \; -1/5 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t} + 1/5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} \right) \\ &= \begin{pmatrix} 1/5\left(e^{-3t} + 4e^{2t}\right) & 1/5\left(-e^{-3t} + e^{2t}\right) \\ 4/5\left(-e^{-3t} + e^{2t}\right) & 1/5\left(4e^{-3t} + e^{2t}\right) \end{pmatrix} \end{aligned}$$

c) Find the solution to the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ corresponding to the initial conditions $\mathbf{y}(t = 0) = \mathbf{y^0}$:
i) $\mathbf{y^0} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, ii) $\mathbf{y^0} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ iii) $\mathbf{y^0} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$ iv) $\mathbf{y^0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Once we find e^{At}, the problem of solving the system $\mathbf{y}' = A\mathbf{y}$ with a given initial condition reduces to matrix multiplication as the solution is just $\mathbf{y} = e^{At}\mathbf{y_0}$.

For example, if $y_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, we get

$$1/5 \begin{pmatrix} e^{-3t} + 4e^{2t} \\ 4\left(-e^{-3t} + e^{2t}\right) \end{pmatrix} - 1/5 \begin{pmatrix} -e^{-3t} + e^{2t} \\ 4e^{-3t} + e^{2t} \end{pmatrix}$$

If $y_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ then

$$y = 1/5 \begin{pmatrix} e^{-3t} + 4e^{2t} \\ 4\left(-e^{-3t} + e^{2t}\right) \end{pmatrix} + 2/5 \begin{pmatrix} -e^{-3t} + e^{2t} \\ 4e^{-3t} + e^{2t} \end{pmatrix}$$

If $\mathbf{y_0} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, then

$$y = 2/5 \begin{pmatrix} e^{-3t} + 4e^{2t} \\ 4\left(-e^{-3t} + e^{2t}\right) \end{pmatrix} + \begin{pmatrix} -e^{-3t} + e^{2t} \\ 4e^{-3t} + e^{2t} \end{pmatrix}$$

Finally if $\mathbf{y_0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then we just take the second column of e^{At}:

$$y = 1/5 \begin{pmatrix} -e^{-3t} + e^{2t} \\ 4e^{-3t} + e^{2t} \end{pmatrix}$$

At this point, you may see the advantage of computing the special fundamental matrix - if you need to solve the same problem for many initial conditions!

5) a) Given a matrix $\mathbf{A}$, show that the characteristic equation determining its eigenvalues may be written as

$$\lambda^2 - tr\left(\mathbf{A}\right)\lambda + \det \mathbf{A} = 0$$

(where $tr\left(\mathbf{A}\right)$ is the trace and $\det \mathbf{A}$ the determinant of $\mathbf{A}$)
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the eigenvalues are found by solving

$$\begin{aligned} \det\left(A - \lambda I\right) &= \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= \lambda^2 - \left(a+b\right)\lambda + ab - cd \end{aligned}$$

Now we recognize that $tr(A) = a + b$ is the sum of diagonal elements of A while $\det\left(A\right) = ab - cd$
b) Suppose that the trace and determinant of $\mathbf{A}$ are both positive. Show that the phase portrait of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is either an unstable node or an unstable spiral. (and no other case is possible) (hint: use part a) of course!)
Solution: Using part a), we have that

$$\lambda = \frac{1}{2}\left(tr\left(A\right) \pm \sqrt{\left(tr\left(A\right)\right)^2 - 4\det A}\right)$$

If $tr(A) > 0$ and $\det(A) > 0$, the result follows immediately: either we have an unstable spiral (when the argument of the square root is negative) and the real part of λ is larger than zero. Else, in the worst case,

$$\lambda = \frac{1}{2}\left(tr(A) - \sqrt{\left(tr\left(A\right)\right)^2 - 4\det A}\right) > 0$$

since $tr(A) > \sqrt{tr(A)^2 - 4\det A}$ whenever $\det(A) > 0$.

c) Let $\mathbf{A}$ be given by

$$\begin{pmatrix} \alpha & -2 \\ \beta & 3 \end{pmatrix}$$

where α and β are positive. Find a condition on α, β that result in an unstable node or unstable spiral. Draw an example phase portrait in each case. What happens when $\beta = \frac{1}{8}(\alpha - 3)^2$? (Hint: Use part b)!)

Here $tr(A) = \alpha + 3$ and $\det(A) = 3\alpha + 2\beta$. The case separating the spiral from the node is the sign of the expression under the square root. That is we have a node if

$$\begin{aligned} (tr(A))^2 &> 4\det(A) \\ (\alpha+3)^2 &> 4(3\alpha + 2\beta) \\ \alpha^2 + 6\alpha + 9 &> 12\alpha + 8\beta \\ \alpha^2 - 6\alpha + 9 &> 8\beta \\ (\alpha - 3)^2 &> 8\beta \end{aligned}$$

And we have the spiral if

$$(\alpha - 3)^2 < 8\beta$$

In the borderline case $\beta = \frac{1}{8}(\alpha - 3)^2$, there is only 1 eigenvalue and we typically see an improper node.