Tutorial Problems #3

MAT 267 – Advanced Ordinary Differential Equations – Fall 2014

Christopher J. Adkins

Solutions

pg.109 - # 2 - Petrov Solve

$$\begin{cases} xy_1' = 2y_1 - y_2 \\ xy_2' = 2y_1 - y_2 \end{cases}$$

(a) Show if $x_0 \neq 0$, the solution exists and is unique on the real axis and if $x_0 = 0$, the solution exists only if $2y_1 - y_2 = 0$ and is not unique.

(b) Show the Wronskian of the linearity independent solutions is Cx with $C \neq 0$,

Solution We'll first solve the system. Notice that

$$xy_1' = xy_2' \implies y_1' = y_2' \quad \text{when} \quad x \neq 0$$

Thus $y_1 = y_2 + C_1$ with some constant $C_1 \in \mathbb{R}$. Using this, we see the system reduces to

$$xy_1' = 2y_1 - y_1 - C_1 = y_1 - C_1$$

This equation is separable, thus

$$\int \frac{dy_1}{y_1 - C_1} = \int \frac{dx}{x} \implies \ln|y_1 - C_1| = \ln|x| + C \implies y_1 = C_1 + C_2 x$$

Now that we have y_1 it's easy to see that

$$y_2 = 2C_1 + C_2 x$$

You may write this in vector notation as

$$\boxed{y(x) = C_1 y^{(1)} + C_2 y^{(2)} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 x \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

We compute the Wronskian by definition:

$$W(x) = \det(y^{(1)} y^{(2)}) = \begin{vmatrix} 1 & x \\ 2 & x \end{vmatrix} = Cx \quad \text{where} \quad C \neq 0$$

Notice that $x_0 \neq 0$ implies the Wronskian is non-zero as long as x remains on x_0's side of zero, hence the two solutions we found are linearly independent and unique. If $x_0 = 0$, then $W(x) = 0$ since it is either always non-zero or zero, we know the solutions cannot be linearly independent, i.e. $y_1 = ay_2$ with some $a \in \mathbb{R}$. But this means that we need $xy_1' = xy_2' = axy_1'$ which implies that $2y_1 - y_2 = 0$ for all x. So

$$y_1 = C \quad \& \quad y_2 = 2C \quad \text{where} \quad C \in \mathbb{R}$$

□

n-th order ODE's as first order systems Notice that we have

$$y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_0 y^{(0)} = 0 \iff \dot{x} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & & 1 \\ -p_0 & -p_1 & \ldots & \ldots & -p_{n-1} \end{pmatrix} x \quad \text{where} \quad x = \begin{pmatrix} y \\ y' \\ \vdots \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

pg.770 - # 6 Set up a system of first order equations for

$$y''' = 2x(y')^2 - 3yy'' + xy, \quad y(0) = 1, y'(0) = -1, y''(0) = 2$$

Solution Start from the top and let

$$y_1 = y \quad y_2 = y_1' = y' \quad y_3 = y_1'' = y_2' = y''$$

Then we see that

$$y_3' = y''' = 2xy_2^2 - 4y_1y_3 + xy_1, \quad y_1(0) = 1, y_2(0) = -1, y_3(0) = 2$$

or in matrix notation we have that

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2xy_2^2 - 4y_1y_3 \end{pmatrix}$$

□

Picard Iterations for first order systems Suppose that

$$\begin{cases} \dot{x} = F(t, x(t)) \\ x(t_0) = x_0 \end{cases} \quad x \in \mathbb{R}^n, \quad F(t,x) : \mathbb{R} \times C[\mathbb{R}]^n \to \mathbb{R}^n$$

Then we still have the fundamental theorem of calculus element wise to conclude Picard iterations of the form

$$\phi_0 = x_0 \quad \& \quad \phi_{k+1} = x_0 + \int_{t_0}^{t} F(s, x(s))ds$$

where the integral is element wise. Thus the previous existence and uniqueness proof follows if $F(t,x)$ has Lipschitz functions.

pg.726 - # 9 Find the first few Picard iterates for

$$\frac{dx}{dt} = y^2, \frac{dy}{dt} = x + z, \frac{dz}{dt} = z - y, \quad x(0) = 1, y(0) = 0, z(0) = 1$$

Solution Note that we may rewrite the above as

$$\dot{x} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} x + \begin{pmatrix} y^2 \\ 0 \\ 0 \end{pmatrix} \qquad x(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Using the above formula for Picard iterations we see

$$\phi_0 = x_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\phi_1 = x_0 + \int_0^t F(s, \phi_0) ds = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 2t \\ 1+t \end{pmatrix}$$

$$\phi_2 = x_0 + \int_0^t F(s, \phi_1) ds = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 4s^2 \\ 2+s \\ 1-s \end{pmatrix} ds = \begin{pmatrix} 1 + 4t^3/3 \\ 2t + t^2/2 \\ t - t^2/2 \end{pmatrix}$$

□

A Helpful Formula to Remember Liouville's Formula. Let X be the fundamental solution to $\dot{X} = AX$ with $X(x_0) = X_0$, then you have

$$\det X(x) = \det X_0 \exp\left(\int_{x_0}^x \operatorname{tr}(A(s)) ds\right)$$

Abel's Formula for the Wronskian of n-th order ODE is now an easy corollary. If $y_1, \ldots, y_n$ solve

$$y^{(n)} + p_{n-1} y^{(n-1)} + \ldots + p_0 y^{(0)} = 0$$

The Wronskian for the solutions is given by

$$W[y_1, \ldots, y_n](x) = C \exp\left(\int p_{n-1}(x) dx\right)$$

Quiz Question Prove that if $\phi(0) = 0$ and $\phi'(0)$ exists (and $\phi(x) > 0$), then

$$\int_0^\epsilon \frac{du}{\phi(u)} = \infty \quad \text{for any} \quad \epsilon > 0$$

Solution Since $\phi'(0)$ exists, we see that

$$\phi'(0) = \lim_{h \to 0} \frac{\phi(h) - \phi(0)}{h} = \lim_{h \to 0} \frac{\phi(h)}{h} \in \mathbb{R}$$

Thus we have that ϕ has leading order

$$\phi(x) \approx cx^n \quad n \geqslant 1 \quad c \in \mathbb{R} \setminus \{0\}$$

around 0. Fix $\epsilon > 0$, and take $0 \approx \sigma << \epsilon$. Now by linearity of the integral we may decompose the integral into two pieces, specifically

$$\int_0^\epsilon \frac{du}{\phi(u)} = \int_0^\sigma \frac{du}{\phi(u)} + \int_\sigma^\epsilon \frac{du}{\phi(u)} \geqslant \int_0^\sigma \frac{du}{\phi(u)}$$

By limit comparison, we see

$$\int_0^\sigma \frac{dx}{cx^n} = \infty \implies \int_0^\sigma \frac{du}{\phi(u)} = \infty \implies \int_0^\epsilon \frac{du}{\phi(u)} = \infty$$

□