Tutorial Problems #12

MAT 292 – Calculus III – Fall 2014

Solutions

5.7 - # 15 Consider the initial value problem

$$y'' + \gamma y' + y = k\delta(t-1), \quad y(0) = 0, \quad y'(0) = 0$$

where $|k|$ is the magnitude of an impulse at $t = 1$ and γ is the damping coefficient (or resistance)

(a) Let $\gamma = 1/2$. Find the value of k for which the response has a peak value of 2; call this value k_1

Let's solve the ODE to find the peak value in terms of k. Take the Laplace transform to obtain

$$\mathcal{L}\{y''\} + \gamma\mathcal{L}\{y'\} + \mathcal{L}\{y\} = k\mathcal{L}\{\delta(t-1)\} \implies \mathcal{L}\{y\}(s^2 + \gamma s + 1) = ke^{-s}$$

$$\therefore \mathcal{L}\{y\} = \frac{ke^{-s}}{s^2 + \gamma s + 1}$$

If we complete the square we see

$$\mathcal{L}\{y\} = \frac{ke^{-s}}{\sqrt{1 - \gamma^2/4}} \frac{\sqrt{1 - \gamma^2/4}}{(s + \gamma/2)^2 + 1 - \gamma^2/4}$$

Recall that

$$\mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

Clearly if $\gamma = 1/2$ we have

$$a = -\frac{1}{4} \quad \& \quad b = \frac{\sqrt{15}}{4}$$

Thus we see

$$y(t) = \frac{4k}{\sqrt{15}} e^{-(t-1)/4} \sin\left(\frac{\sqrt{15}}{4}(t-1)\right) u_1(t)$$

So the response should max out around

$$y_{max} = 2 \approx \frac{4k}{\sqrt{15}} e^{-\pi/8} \implies k_1 \approx \frac{\sqrt{15}}{2} e^{\pi/8}$$

(b) Repeat a) for $\gamma = 1/4$ Following the previous question, we see this implies

$$a = -\frac{1}{8} \quad \& \quad b = \frac{\sqrt{63}}{8}$$

Thus...

$$y_{max} = 2 \approx \frac{8k}{\sqrt{63}} e^{-\pi/16} \implies k_1 \approx \frac{\sqrt{63}}{4} e^{\pi/16}$$

(c) Determine how k_1 varies as γ decreases. What is the value of k_1 when $\gamma = 0$?

We see that k_1 decreases as γ decreases. The value of k_1 when $\gamma = 0$ is just the case when

$$a = 0 \quad \& \quad b = 1$$

Thus

$$y_{max} = 2 = k \implies k_1 = 2$$

5.7 - #25b Show that if $f(t) = \delta(t - \pi)$ then

$$y(t) = \int_0^t e^{-(t-\tau)} f(\tau) \sin(t - \tau)) d\tau$$

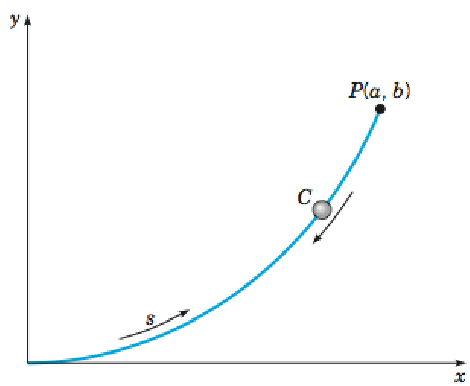
reduces to $y = u_\pi(t) e^{-(t-\pi)} \sin(t - \pi)$.

Solution Plug and chug

$$\begin{aligned} y(t) &= \int_0^t e^{-(t-\tau)} \delta(t - \pi) \sin(t - \tau)) d\tau \\ &= \begin{cases} 0 & t < \pi \\ e^{-(t-\pi)} \sin(t - \pi) & t \geqslant \pi \end{cases} \\ &= u_\pi(t) e^{-(t-\pi)} \sin(t - \pi) \end{aligned}$$

□

5.8- # 31 A problem of interest in the history of mathematics is that of finding the tautochrone the curve down which a particle will slide freely under gravity alone, reaching the bottom in the same time regardless of its start- ing point on the curve. This problem arose in the construction of a clock pendulum whose period is independent of the amplitude of its motion. The tautochrone was found by Christian Huygens in 1673 by geometrical methods, and later by Leibniz and Jakob Bernoulli using analytical arguments. Bernoullis solution (in 1690) was one of the first occasions on which a differential equation was explicitly solved. The geometric configuration is shown in figure.



The starting point $P(a, b)$ is joined to the terminal point $(0, 0)$ by the arc C. Arc length s is measured from the origin, and $f(y)$ denotes the rate of change of s with respect to y:

$$f(y) = \frac{ds}{dy} = \left[1 + \left(\frac{dx}{dy}\right)^2\right]^{1/2}$$

Then it follows from the principle of conservation of energy that the time $T(b)$ required for a particle to slide from P to the origin is

$$T(b) = \frac{1}{\sqrt{2g}} \int_0^b \frac{f(y)}{\sqrt{b-y}} dy$$

(a) Assume that $T(b) = T_0$, a constant, for each b. By taking the Laplace transform of $T(b)$ in this case, and using the convolution theorem, show that

$$F(s) = \sqrt{\frac{2g}{\pi}} \frac{T_0}{\sqrt{s}}$$

then show that

$$f(y) = \frac{\sqrt{2g}}{\pi} \frac{T_0}{\sqrt{y}}$$

¶Recall that

$$f * g(t) = \int_0^t f(t-\tau) g(\tau) d\tau \quad \& \quad \mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$

Thus using the formula for $T(b)$ we obtain (note $T(b) = T_0$)

$$\mathcal{L}\{\sqrt{2g} T_0\} = \mathcal{L}\left\{ f * \frac{1}{\sqrt{t}} \right\} \implies \frac{\sqrt{2g} T_0}{s} = \mathcal{L}\{f\}\mathcal{L}\left\{ \frac{1}{\sqrt{t}} \right\}$$

Recall that $\mathcal{L}\left\{ \frac{1}{\sqrt{t}} \right\} = \sqrt{\pi/s}$, we computed this 2 weeks ago. Thus

$$\mathcal{L}\{f\} = \sqrt{\frac{2g}{\pi s}} T_0$$

If we compute the inverse by comparison with $\mathcal{L}\left\{ \frac{1}{\sqrt{t}} \right\} = \sqrt{\pi/s}$, we obtain

$$f(y) = \frac{\sqrt{2g}}{\pi} \frac{T_0}{\sqrt{y}}$$

(b) Combining the previous equations, show that

$$\frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}}$$

where $\alpha = gT_0^2/\pi^2$.

¶We have that

$$f(y) = \frac{ds}{dy} = \left[1 + \left(\frac{dx}{dy} \right)^2 \right]^{1/2} = \frac{\sqrt{2g}}{\pi} \frac{T_0}{\sqrt{y}}$$

Thus

$$1 + \left(\frac{dx}{dy} \right)^2 = \frac{2g}{\pi^2} \frac{T_0^2}{y} \implies \left(\frac{dx}{dy} \right)^2 = \frac{2\alpha}{y} - 1 \implies \frac{dx}{dy} = \sqrt{\frac{2\alpha - y}{y}}$$

(c) Use the subsitution $y = 2\alpha \sin^2(\theta/2)$ to solve the previous ODE, and show

$$x = \alpha(\theta + \sin\theta), \quad y = \alpha(1 - \cos\theta)$$

These equations can be identified as parametric equations of a cycloid. Thus the tautochrone is an arc of a cycloid.

¶If we plug in the chosen y and using the fact $1-\sin^2\theta = \cos^2\theta$, we see

$$\frac{dx}{dy} = \frac{1}{2\alpha\sin(\theta/2)\cos(\theta/2)}\frac{dx}{d\theta} = \cot(\theta/2) \implies \frac{dx}{d\theta} = 2\alpha\cos^2(\theta/2)$$

This equation is separable, thus

$$x(\theta) = 2\alpha\int\cos^2(\theta/2)d\theta = \alpha\int(1+\cos\theta)d\theta = \alpha(\theta+\sin\theta)$$

via the half-angle identity. By choice of y, the half-angle identity also gives

$$y(\theta) = 2\alpha\sin^2(\theta/2) = \alpha(1-\cos(\theta))$$

9.5 - # 1 Determine whether separation of variables can be used in the PDE. If so, find the separated ODE's.

$$xu_{xx} + u_t = 0$$

Solution We see the answer is yes here since if $u(x,t) = X(x)T(t)$, we obtain

$$xX''T + XT' = 0 \implies \frac{xX''}{X} + \frac{T'}{T} = 0 \implies \underbrace{\frac{xX''}{X}}_{f(x)} = -\underbrace{\frac{T'}{T}}_{g(t)}$$

since both functions (f and g) do not depend on each other, they must be constant. Call the constant $\lambda\in\mathbb{R}$, thus

$$\boxed{xX'' = \lambda X \quad \& \quad T' = -\lambda T}$$

are the ODE's we seek.

9.5 - # 3 Determine whether separation of variables can be used in the PDE. If so, find the separated ODE's.

$$u_{xx} + u_{xt} + u_t = 0$$

Solution We again see the answer is yes here. If we try $u(x,t) = X(x)T(t)$, we'll obtain

$$\frac{X''}{X} + \frac{X'T'}{XT} + \frac{T'}{T} == \frac{X''}{X} + \frac{T'}{T}\left(\frac{X'}{X}+1\right) = 0 \implies \frac{X''}{X'+X} = -\frac{T'}{T}$$

thus we see the above both equal some constant $\lambda\in\mathbb{R}$, hence

$$\boxed{X'' = \lambda(X'+X) \quad \& \quad T' = -\lambda T}$$

are the ODE's we seek.

9.5 - # 5 Determine whether separation of variables can be used in the PDE. If so, find the separated ODE's.

$$u_{xx} + (x+y)u_{yy} = 0$$

Solution We won't be able to decouple the system due to the mixed variables. If we try $u(x,y) = X(x)Y(y)$, we see

$$\frac{X''}{X} + (x+y)\frac{Y''}{Y} = 0$$

which can not be decoupled.