Tutorial Problems #3

MAT 292 – Calculus III – Fall 2014

Solutions

2.3 - # 22 Verify the both $y_1(t) = 1 - t$ and $y_2(t) = -t^2/4$ are solutions of the initial value problem

$$y' = \frac{-t + (t^2 + 4y)^{1/2}}{2}, \qquad y(2) = -1$$

Where are these solutions valid? Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Picard-Lindelöf Theorem.

Solution To check if these are solutions to the IVP, we compute the derivative and see if equality holds true. For y_1 we have

$$LHS = y_1' = -1, \quad RHS = \frac{-t + \sqrt{(t^2 + 4 - 4t)}}{2} = \frac{-t + \sqrt{(t-2)^2}}{2} = \frac{-t + |t-2|}{2} = \begin{cases} -1 & t \geqslant 2 \\ 1 - t & t < 2 \end{cases}$$

For y_2 we have

$$LHS = y_2' = -\frac{t}{2}, \quad RHS = \frac{-t + \sqrt{(t^2 - t^2)}}{2} = \frac{-t}{2}$$

We see y_1 solves the problem on $[2, \infty)$ and y_2 on $\mathbb{R}$. This doesn't violate the existence-uniqueness theorem since

$$f(t,y) = \frac{-t + (t^2 + 4y)^{1/2}}{2} \implies \frac{\partial f}{\partial y} = \frac{1}{\sqrt{t^2 + 4y}} \implies \frac{\partial f}{\partial y} \quad \text{isn't continuous(or Lipschitz) around } (2, -1)$$

Thus the Picard-Lindel"of Theorem does not apply. $\square$

Example Consider

$$\begin{cases} y' = -y^2 \\ y(0) = -1 \end{cases}$$

Show the solution is unique and exists, then solve the system. Where is the domain of the solution?

Solution Well, $y' = f(t, y)$, thus in our case we have

$$f(t,y) = -y^2 \implies \frac{\partial f}{\partial y} = -2y$$

which is continuous. Thus the solution is unique and exists by the Picard-Lindelöf Theorem. The solution to the ODE can be found using separation of variables, i.e.

$$y' = -y^2 \implies \int \frac{dy}{y^2} = \int -dt \implies \frac{1}{y} = t + C \implies \frac{1}{y} = t - 1 \implies y(t) = \frac{1}{t-1}$$

The domain is seen to be $(-\infty, 1)$ since we need to satisfy $0 \in (-\infty, 1)$. □

2.3 - # 32 Solve

$$\begin{cases} y' + 2y = g(t) \\ y(0) = 0 \end{cases}, \quad \text{where} \quad g(t) = \begin{cases} 1 & 0 \leqslant t \leqslant 1 \\ 0 & t > 1 \end{cases}$$

Solution We know

$$y(t) = \frac{1}{\mu} \int \mu g dt \quad \text{where} \quad \mu = \exp\left[\int p(t)dt\right] = e^{2t}$$

Thus the solution on $[0, 1]$ is given by

$$y(t) = e^{-2t} \int_0^t e^{2s} ds = \frac{1 - e^{-2t}}{2}$$

The solution on $t > 1$ is given by

$$y(t) = Ce^{-2t}$$

Since we'd like a continuous solution, we find C by setting both pieces equal to each other at $t = 1$.

$$Ce^{-2} = \frac{1 - e^{-2}}{2} \implies C = \frac{e^2 - 1}{2}$$

Thus a piecewise solution to the ODE is given by

$$y(t) = \begin{cases} \frac{1-e^{-2t}}{2} & 0 \leqslant t \leqslant 1 \\ \frac{e^{2(1-t)} - e^{-2t}}{2} & t > 1 \end{cases}$$

□

2.3 - #34 Consider

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

Write this in definite form. Using the definite form, assume $p \geqslant p_0 > 0$ for all $t \geqslant t_0$ and $|g(t)| \leqslant M$ for all $t \geqslant t_0$. Show that $y(t)$ is bounded. Also construct an example.

Solution To do this, let's define our integrating factor to satisfy $\mu(t_0) = 1$, i.e.

$$\mu(t) = \exp\left[\int_{t_0}^t p(s)ds\right]$$

Thus we may write

$$y(t) = \frac{1}{\mu(t)} \left[\int_{t_0}^t \mu(s)g(s)ds + y(t_0)\right]$$

One may combine the integrating factor into the integral by recalling that

$$\int_t^{t_0} f(\xi)d\xi + \int_{t_0}^s f(\xi)d\xi = \int_t^s f(\xi)d\xi$$

and the additivity of the exponential. i.e.

$$\frac{1}{\mu(t)}\left[\int_{t_0}^{t}\mu(s)g(s)ds + y(t_0)\right] = \int_{t_0}^{t}\frac{\mu(s)}{\mu(t)}g(s) + \frac{y(t_0)}{\mu(t)}$$

where

$$\frac{\mu(s)}{\mu(t)} = \frac{\exp\left[\int_{t_0}^{s}p(\xi)d\xi\right]}{\exp\left[\int_{t_0}^{t}p(\xi)d\xi\right]} = \exp\left[\int_{t_0}^{s}p(\xi)d\xi - \int_{t_0}^{t}p(\xi)d\xi\right] = \exp\left[-\int_{s}^{t}p(\xi)d\xi\right]$$

Thus we may write

$$\boxed{y(t) = \int_{t_0}^{t}g(s)\exp\left[-\int_{s}^{t}p(\xi)d\xi\right]ds + y(t_0)\exp\left[-\int_{t_0}^{t}p(\xi)d\xi\right]}$$

Now if we assume $p \geqslant p_0 > 0$ and $|g(t)| \leqslant M$, we easily have that $-p \leqslant -p_0 < 0$, thus

$$\exp\left[\int_{t_0}^{t}-p(\xi)d\xi\right] \leqslant \exp\left[\int_{t_0}^{t}-p_0 d\xi\right] = e^{-p_0(t-t_0)}$$

This bound shows the second terms is bounded by

$$\left|y(t_0)\exp\left[-\int_{t_0}^{t}p(\xi)d\xi\right]\right| \leqslant |y(t_0)|, \quad \forall t \geqslant t_0$$

To bound the first term, notice that

$$\left|\int_{t_0}^{t}g(s)\exp\left[-\int_{s}^{t}p(\xi)d\xi\right]ds\right| \leqslant M\int_{t_0}^{t}\exp\left[-\int_{s}^{t}p(\xi)d\xi\right]ds$$

since $|g| \leqslant M$. Using the previous bound, we see that

$$M\int_{t_0}^{t}\exp\left[-\int_{s}^{t}p(\xi)d\xi\right]ds \leqslant M\int_{t_0}^{t}e^{-p_0(t-s)}ds = Me^{-p_0 t}\int_{t_0}^{t}e^{p_0 s}ds = \frac{M(1-e^{-p_0(t-t_0)})}{p_0} \leqslant \frac{M}{p_0}$$

Putting both bounds together with the triangle inequality we see

$$\boxed{|y(t)| \leqslant \frac{M}{p_0} + |y(t_0)|, \quad \forall t \geqslant t_0}$$

For an example, take $g(t) = e^{-t}(\leqslant 1)$, $y(0) = y_0$ and $p(t) = 1(\geqslant 1)$.

Picard Iterations Consider

$$y' = f(t, y(t)), \quad y(t_0) = y_0$$

Describe how to do the Picard Iteration Method here. Then assume $\phi_n \to \phi$, and show that ϕ is a solution of the IVP.

Solution Notice by the fundamental theorem of Calculus we may write

$$y(t) = y(t_0) + \int_{t_0}^{t}f(s, y(s))ds$$

To approximate the answer to this integral equation, take the approximating sequence

$$\phi_0 = y_0, \quad \phi_{k+1} = y_0 + \int_{t_0}^{t}f(s, \phi_k(s))ds$$

This is why the Picard Iteration works if f is nice enough. If we assume that our approximation converges to some answer ϕ, we see this implies that

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds$$

This solves the IVP by differentiating the integral equation. $\square$