Tutorial Problems #5

MAT 292 – Calculus III – Fall 2014

Solutions

1.

(a) The equation for v is:

$$v' = f(t, x, v), \quad v(t_0) = v_0.$$

For $t_1 = t_0 + h$ we have by Euler's method that:

$$v_1 = v_0 + h \cdot f(t_0, x_0, v_0). \tag{0.1}$$

(b) Heun's (or improved Euler's) method for $x(t)$ is

$$x_{n+1} = x_n + \frac{k_{n,1} + k_{n,2}}{2} h,$$

where

$$k_{n,1} \approx x'(t_n) \approx v_n$$
$$k_{n,2} \approx x'(t_{n+1}) \approx v_{n+1}$$

So we write it as

$$x_{n+1} = x_n + \frac{v_n + v_{n+1}}{2} h. \tag{0.2}$$

We can compute v_{n+1} using **(a)** before we compute x_{n+1}.

c) By formulas (0.1) and (0.2) we can easily see that we have:

$$v_{n+1} = v_n + h \cdot f(t_n, x_n, v_n),$$

and

$$x_{n+1} = x_n + \frac{v_n + v_{n+1}}{2} h,$$

where $t_{n+1} = t_n + h$, for all $n \in \mathbb{N}$.

2.7.17. Note that we are looking for the *local truncation error*, so we assume that the (t_n, y_n) is equal to the solution: $\phi(t_n) = y_n$.

(a) By Taylor's formula (stated in the exercise) and since $\phi'(t_n) = f(t_n, x_n) := f_n$ we have that:

$$\begin{aligned} e_{n+1} &= [\phi(t_n) - y_n] + \left[\phi'(t_n) - \frac{f_n}{2}\right] \cdot h + \left[\phi''(t_n) \cdot \frac{h^2}{2!} - \frac{f(t_n + h, y_n + hf_n)}{2} \cdot h\right] + \phi'''(\bar{t_n})\frac{h^3}{3!} \\ &= \frac{\phi''(t_n) \cdot h + f_n - f(t_n + h, y_n + hf_n)}{2} \cdot h + \phi'''(\bar{t_n})\frac{h^3}{3!}, \end{aligned}$$

for some $\bar{t_n} \in (t_n, t_{n+1})$.

(b) By Taylor's formula for functions of two variables (also stated in the exercise) we have that:

$$\begin{gathered} f(t_n + h, y_n + hf_n) - f(t_n, y_n) = f_t(t_n, y_n) \cdot h + f_y(t_n, y_n) \cdot hf_n + \\ + \frac{1}{2!}\left(h^2 f_{tt}(\bar{t_n}, \bar{y_n}) + 2h^2 f_n f_{ty}(\bar{t_n}, \bar{y_n}) + h^2 f_n^2 f_{yy}(\bar{t_n}, \bar{y_n})\right), \end{gathered}$$

for $\bar{t_n} \in (t_n, t_{n+1})$ and $\bar{y_n} \in (y_n, y_n + hf_n)$.

Going back to the equation of part **(a)**, we see that since

$$\phi''(t_n) = f_t(t_n, y_n) + f_y(t_n, y_n) \cdot f_n,$$

we finally have that

$$e_{n+1} = \frac{1}{2!}\left(h^2 f_{tt}(\bar{t_n}, \bar{y_n}) + 2h^2 f_n f_{ty}(\bar{t_n}, \bar{y_n}) + h^2 f_n^2 f_{yy}(\bar{t_n}, \bar{y_n})\right) \cdot h + \phi'''(\bar{t_n})\frac{h^3}{3!}, \tag{0.3}$$

which is a cubic polynomial with respect to h.

(c) Since we have that:

$$f(t, y) = at + by + c,$$

for constants a, b, c, we can easily check that

$$f_{tt} = f_{ty} = f_{yy} \equiv 0.$$

Now it follows from (0.3) that

$$e_{n+1} = \phi'''(\bar{t_n})\frac{h^3}{3!}$$

in this case.

Note. In fact, we can write equation (0.3) as

$$e_{n+1} = \frac{1}{2}\left(f_{tt}(\bar{t_n}, \bar{y_n}) + 2f_n f_{ty}(\bar{t_n}, \bar{y_n}) + f_n^2 f_{yy}(\bar{t_n}, \bar{y_n})\right) \cdot h^3 + \phi'''(\bar{t_n})\frac{h^3}{3!}.$$

This means that as long as the function f has continuous third partial derivatives, then we can say that

$$e_{n+1} \leqslant Ch^3.$$

3.

(a) For a partition of the time axis $\{t_n\}_{n\in\mathbb{N}}$, we have that:

$$x'(t_n) \approx \frac{x(t_n) - x(t_{n-1})}{t_n - t_{n-1}}. \tag{0.4}$$

(b) We have that:

$$f(t_n, x(t_n), x'(t_n)) \approx \frac{x'(t_{n+1}) - x'(t_n)}{t_{n+1} - t_n},$$

which by (0.4) becomes:

$$f(t_n, x(t_n), x'(t_n)) \approx \frac{x(t_{n+1}) - x(t_n)}{(t_{n+1} - t_n)^2} - \frac{x(t_n) - x(t_{n-1})}{(t_{n+1} - t_n)(t_n - t_{n-1})}. \tag{0.5}$$

(c) Consider $t_{n+1} = t_n + h$ for some step $h > 0$. Then from **(a)** and **(b)** we get

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} = f(t_n, x_n, v_n),$$

where $v_n = \dfrac{x_n - x_{n-1}}{h}$. We then have the numerical scheme

$$x_{n+1} = 2x_n - x_{n-1} + f(t_n, x_n, v_n)h^2 \qquad \text{for } n \geqslant 1.$$

(d) The numerical scheme in **(c)** skips x_1, so we need to obtain x_1 in a different way. We can use Euler's method:

$$x_1 = x_0 + v_0 h.$$