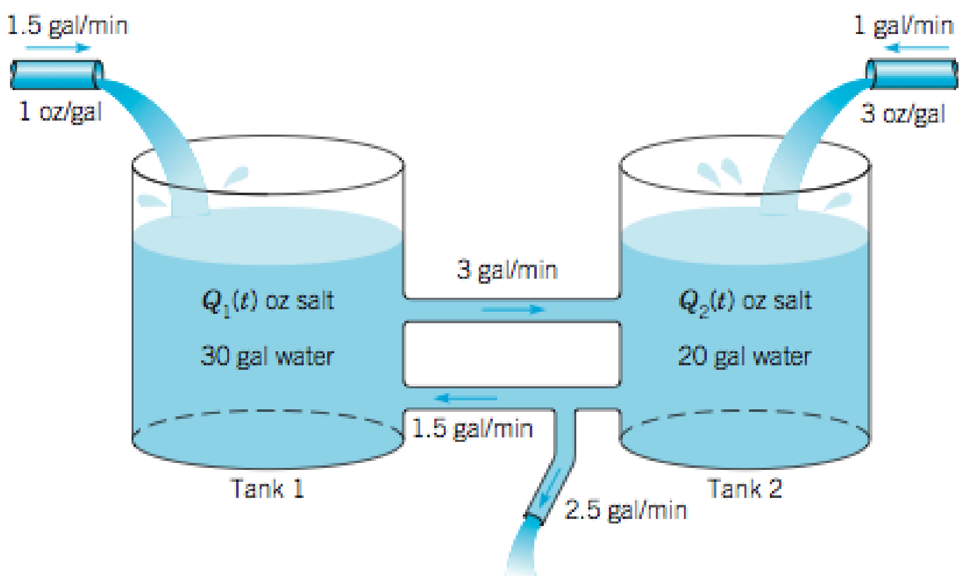
Tutorial Problems #6

MAT 292 – Calculus III – Fall 2014

SOLUTIONS

3.2 - # 30 A Mixing Problem. Each of the tanks shown in the figure contains a brine solution. Assume that Tank 1 initially contains 30 gal of water and 55 oz of salt, and Tank 2 initially contains 20 gal of water and 26 oz of salt. Water containing 1 oz/gal of salt flows into Tank 1 at a rate of 1.5 gal/min, and the well-stirred solution flows from Tank 1 to Tank 2 at a rate of 3 gal/min. Additionally, water containing 3 oz/gal of salt flows into Tank 2 at a rate of 1 gal/min (from the outside). The well-stirred solution in Tank 2 drains out at a rate of 4 gal/min, of which some flows back into Tank 1 at a rate of 1.5 gal/min, while the remainder leaves the system. Note that the volume of solution in each tank remains constant since the total rates of flow in and out of each tank are the same: 3 gal/min in Tank 1 and 4 gal/min in Tank 2.



(a) Denoting the amount of salt in Tank 1 and Tank 2 by $Q_1(t)$ and $Q_2(t)$, respectively, use the principle of mass balance to show that

$$\frac{dQ_1}{dt} = -0.1Q_1 + 0.075Q_2 + 1.5,$$

$$\frac{dQ_2}{dt} = 0.1Q_1 - 0.2Q_2 + 3$$

$$Q_1(0) = 55, \quad Q_2(0) = 26$$

¶Use the principle of mass balance, i.e.

$$\frac{dQ}{dt} = Q_{in} - Q_{out}$$

Lets do tank 1 first. Clearly we have that Q_1 leaves that tank at a rate of 3 gal/min, so since the volume stays constant (at 30 gal), we have that $Q_{out} = 3/30Q_1 = 0.1Q_1$. The amount coming in is similar, since we have the constant flow of 1.5 oz/min being added, and the flow from Q_2 which is just $1.5/20Q_2 = 0.075Q_2$. Hence

$$\frac{dQ_1}{dt} = -0.1Q_1 + 0.075Q_2 + 1.5$$

The other tank is completely similar.

(b) Write the initial value problem using matrix notation.

¶Let $Q = (Q_1, Q_2)^T$, then we have

$$Q' = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix} Q + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$$

(c) Find the equilibrium value Q_1^E and Q_2^E of the system.

¶Find the critical points($Q' = 0$)! I.e. Solve

$$\begin{pmatrix} 0.1 & -0.075 \\ -0.1 & 0.2 \end{pmatrix} Q^E = \begin{pmatrix} 1.5 \\ 3 \end{pmatrix} \implies Q^E = \begin{pmatrix} 16 & 6 \\ 8 & 8 \end{pmatrix} \begin{pmatrix} 1.5 \\ 3 \end{pmatrix} = \begin{pmatrix} 42 \\ 36 \end{pmatrix}$$

3.3 - # 17-20 Consider $\mathbf{x}' = A\mathbf{x}$. If given the eigenvectors and eigenvalues:

(a) Sketch a phase portrait of the system.

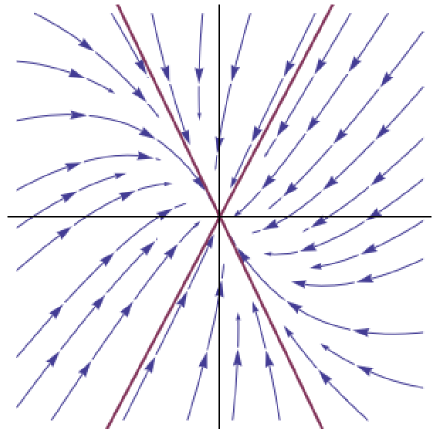
(b) Sketch the trajectory passing through the initial point (2,3)

(c) For the trajectory in part b), sketch the component plots of x_1 versus t and of x_2 versus t on the same set of axes.

 # 17

$$\lambda_1 = -1 \quad \vec{\lambda}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = -2 \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \boxed{x(t) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}}$$

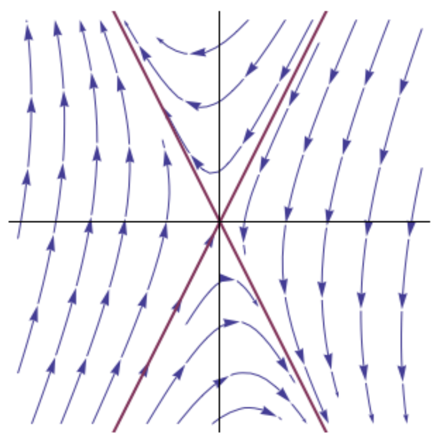
with a portrait like



 # 18

$$\lambda_1 = 1 \quad \vec{\lambda}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = -2 \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \boxed{x(t) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}}$$

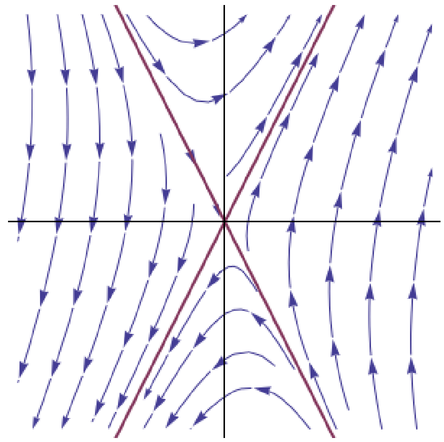
with a portrait like



 # 19

$$\lambda_1 = -1 \quad \vec{\lambda}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = 2 \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \boxed{x(t) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}}$$

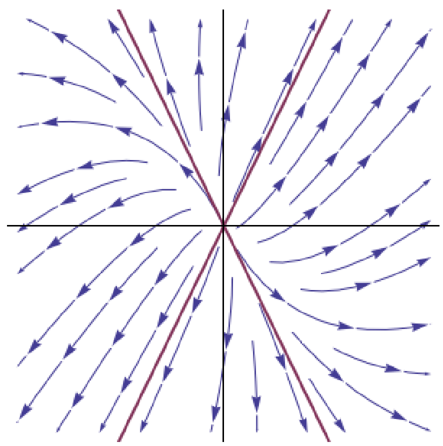
with a portrait like



 # 20

$$\lambda_1 = 1 \quad \vec{\lambda}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad \& \quad \lambda_2 = 2 \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \boxed{x(t) = C_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}}$$

with a portrait like



3.3 - # 26 Consider the system

$$\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ -\alpha & -1 \end{pmatrix} \mathbf{x}$$

(a) Solve the system for $\alpha = 1/2$. What are the eigenvalues of the coefficient matrix? Classify the equilibrium point at the origin as to type.

¶We compute the characteristic equation.

$$P(\lambda) = \det(\mathbb{1}\lambda - A) = \begin{vmatrix} \lambda + 1 & 1 \\ \alpha & \lambda + 1 \end{vmatrix} = \lambda^2 + 2\lambda + 1 - \alpha = 0 \implies \lambda_{\pm} = -1 \pm \sqrt{\alpha}$$

So in this case of $\alpha = 1/2$, we have

$$\lambda_1 = -1 + \frac{1}{\sqrt{2}} \quad \& \quad \lambda_2 = -1 - \frac{1}{\sqrt{2}}$$

Since both eigenvalues are negative and different, this is a node which is asymptotically stable.

(b) How about the case of $\alpha = 2$?

¶Using the above, we clearly see

$$\lambda_1 = -1 + \sqrt{2} \quad \& \quad \lambda_2 = -1 - \sqrt{2}$$

In this case since the eigenvalues are of different signs, we have a saddle.

(c) Notice the change of solutions from a) to b), what α is the critical point when solutions begin to change.

¶By the above, it is when

$$-1 + \sqrt{\alpha} = 0 \implies \alpha = 1$$

3.3 - # 7 Solve the following system, draw direction field and a phase portrait. Describe the behaviour of the solutions as $t \to \infty$

$$\mathbf{x}' = \frac{1}{4}\underbrace{\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}}_{A}\mathbf{x}$$

Solution By now we know the solution is completely characterized by the eigenvalues and eigenvectors of the above matrix. To make the computation nicer, recall that the eigenvalues of A are 4 times what we actually want. Now, let's compute the characteristics equation to find the eigenvalues of A.

$$P(\lambda) = \det(A - \mathbb{1}\lambda) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (\lambda - 8)(\lambda - 2) = 0 \implies \lambda_1 = 8 \,\&\, \lambda_2 = 2$$

Now that we've found the eigenvalues, we must find the eigenvectors! They are easily computed by looking at the kernel of the map evaluated at the eigenvalues

$$\ker(A - \mathbb{1}\lambda_1) = \ker\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \vec{\lambda}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\ker(A - \mathbb{1}\lambda_2) = \ker\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \operatorname{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \implies \vec{\lambda}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

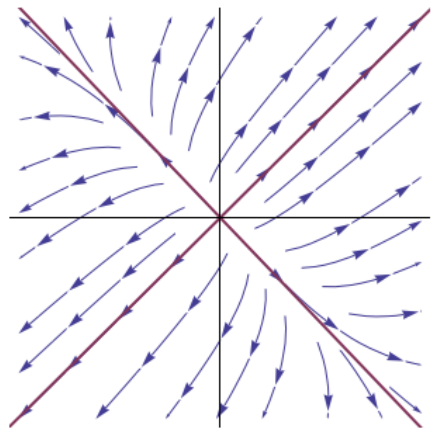
Since eigenvectors are invariant under scaling, we therefore have that actual eigenvalues and eigenvectors are

$$\lambda_1 = 2 \quad \& \quad \vec{\lambda}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \& \quad \lambda_2 = \frac{1}{2} \quad \& \quad \vec{\lambda}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Thus the solution is

$$\boxed{\mathbf{x}(t) = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t/2}}$$

where $C_1, C_2 \in \mathbb{R}$. The system looks like



3.4 - # 7 Solve the following system, draw direction field and a phase portrait. Describe the behaviour of the solutions as $t \to \infty$

$$\mathbf{x}' = \underbrace{\begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}}_{A} \mathbf{x} \qquad \mathbf{x}(0) = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

Solution By now we know the solution is completely characterized by the eigenvalues and eigenvectors of the above matrix. Let's compute the characteristics equation to find the eigenvalues of A.

$$P(\lambda) = \det(\mathbb{1}\lambda - A) = \begin{vmatrix} \lambda+1 & 4 \\ -1 & \lambda+1 \end{vmatrix} = \lambda^2 + 2\lambda + 5 = 0 \implies \lambda_1 = -1 + 2i \quad \& \quad \lambda_2 = -1 - 2i$$

Now that we've found the eigenvalues, we must find the eigenvectors! They are easily computed by looking at the kernel of the map evaluated at the eigenvalues

$$\ker(\mathbb{1}\lambda_1 - A) = \ker \begin{pmatrix} 2i & 4 \\ -1 & 2i \end{pmatrix} = \text{span} \begin{pmatrix} 2i \\ 1 \end{pmatrix} \implies \vec{\lambda}_1 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

$$\ker(\mathbb{1}\lambda_2 - A) = \ker \begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix} = \text{span} \begin{pmatrix} -2i \\ 1 \end{pmatrix} \implies \vec{\lambda}_1 = \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

Thus the solution is

$$x(t) = C_1 \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{(-1+2i)t} + C_2 \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{(-1-2i)t}$$

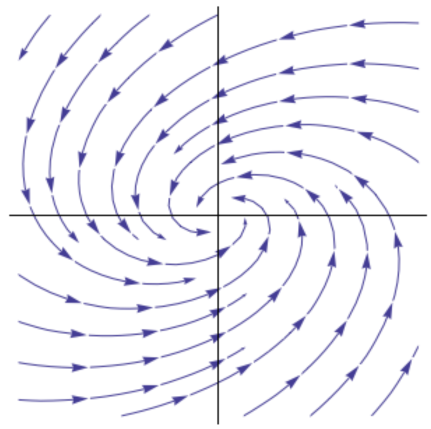
but we'd like a real valued solution, so we call upon the aid of Euler's identity, i.e.

$$e^{i\theta} = \cos\theta + i \sin\theta$$

Using this we obtain

$$\boxed{x(t) = e^{-t}\left[\tilde{C}_1 \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} + \tilde{C}_2 \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix}\right]}$$

where $\tilde{C}_1 = C_1 + C_2$ and $\tilde{C}_2 = i(C_1 - C_2)$. The phase portrait looks like:



3.4 - # 23 In this problem, we indicate how to show that the trajectories are ellipses when the eigenvalues are purely imaginary. Consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$$

(a) Show that the eigenvalues of the coefficient matrix are purely imaginary if and only if

$$a_{11} + a_{22} = 0, \quad a_{11}a_{22} - a_{12}a_{21} > 0$$

¶To show this, we'll compute the characteristic equation and apply the quadratic formula. We have

$$P(\lambda) = \det(A - \mathbb{1}\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - \underbrace{(a_{11} + a_{22})}_{b} \lambda + \underbrace{a_{11}a_{22} - a_{21}a_{12}}_{c} = 0$$

Recall the Quadratic formula for $\lambda^2 + b\lambda + c = 0$,

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Clearly, we need $b = 0$ to kill off the real part, and ensure that $c > 0$. Which is exactly the condition.

(b) The trajectories of the system can be found by converting the system into the single equation

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a_{21}x + a_{22}y}{a_{11}x + a_{12}y}$$

Use the fact that $b = 0$ to show that the above first order equation is exact.

¶We have that

$$\underbrace{(a_{21}x + a_{22}y)}_{M} dx + \underbrace{(-a_{11}x - a_{12}y)}_{N} dy = 0$$

and we need the partials to commute, let's check.

$$M_y = a_{22} \quad \& \quad N_x = -a_{11} \implies M_y = N_x \iff a_{11} + a_{22} = 0$$

Thus the equation is exact.

(c) By solving the exact equation, show that

$$a_{21}x^2 + 2a_{22}xy - a_{12}y^2 = const$$

¶To solve, we integrate each part separately, thus

$$\int M dx = \int (a_{21}x + a_{22}y)dx = \frac{a_{21}}{2}x^2 + a_{22}xy$$

$$\int N dy = \int (-a_{11}x - a_{12}y)dy = -a_{11}xy - \frac{a_{12}}{2}y^2$$

we know that $a_{22} = -a_{11}$, so we have

$$\boxed{a_{21}x^2 + 2a_{22}xy - a_{12}y^2 = const}$$