MAT292 - Calculus III - Fall 2014

Solution of Term Test 1 - October 6, 2014

Time allotted: 90 minutes. Aids permitted: None.

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 16 pages (including this title page). Make sure you have all of them.
- You can use pages 14–16 for rough work or to complete a question (**Mark clearly**).
  DO NOT DETACH PAGES 14–16.

GOOD LUCK!

PART I

No explanation is necessary.

For questions 1–8, consider a constant $a \in \mathbb{R}$ and the differential equation. **(8 marks)**

$$\frac{dy}{dt} = (y+a)(y-a)^2.$$

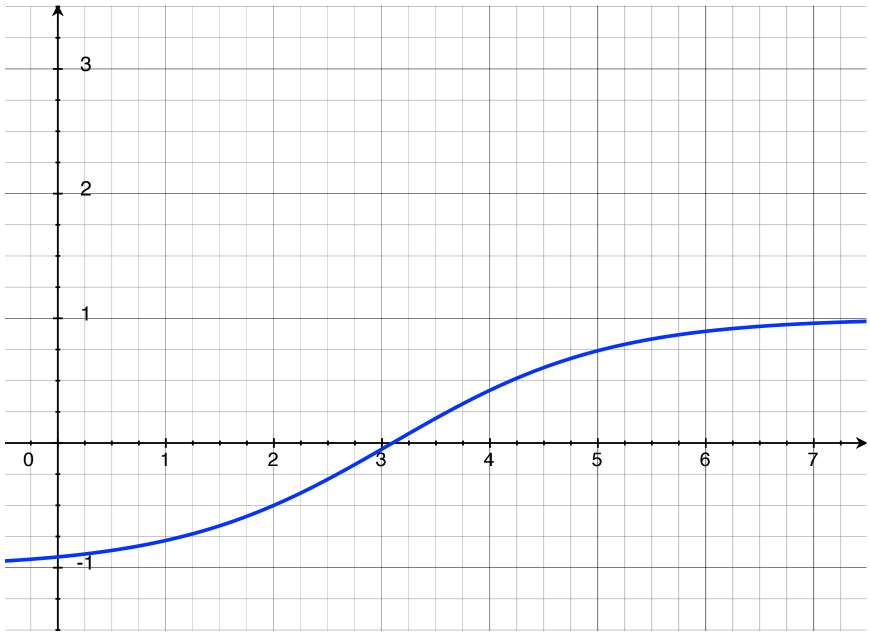
1. If $a > 0$, then the critical point $-a$ is stable / semistable / [✓ unstable]

2. If $a > 0$, then the critical point a is stable / [✓ semistable] / unstable

3. If $a < 0$, then the critical point $-a$ is stable / semistable [✓ unstable]

4. If $a < 0$, then the critical point a is stable / [✓ semistable] / unstable

5. Without solving the differential equation, sketch the solution for $a = 1$ with the initial condition $y(2) = -\frac{1}{2}$.



6. For $a = -1$, the solution has an asymptote at $y = 1$ as $t \to +\infty$ if the initial condition is

(a) $y(42) = 0$

(b) $y(-28) = 2$

(c) $y(0) = -2$

$\boxed{\textbf{(d)}}$ Only the equilibrium solution can have asymptote at $y = 1$.

7. For $a = -1$, the solution has an asymptote at $y = -1$ as $t \to +\infty$ if the initial condition is

$\boxed{\textbf{(a)}}$ $y(10^{10}) = 0$

(a) $y(-4000) = 2$

(b) $y(2014) = -2$

(c) Only the equilibrium solution can have asymptote at $y = -1$.

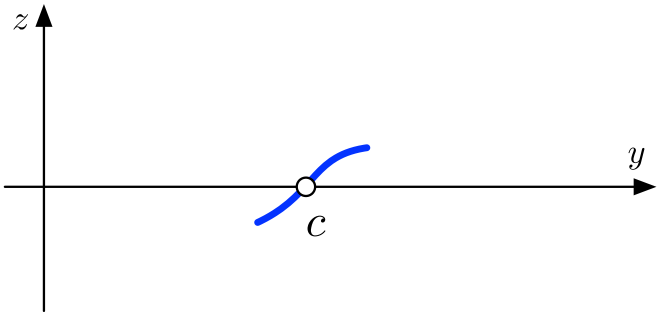
8. Let $a < 0$ and let $y = \phi(t)$ be the solution with initial condition $y(0) = \frac{a}{2}$. Then the maximum of $\phi(t)$ for $t \geqslant 0$ is

$$\max_{t\in[0,\infty)} \phi(t) = \underline{\quad a/2. \qquad\qquad\qquad\qquad}$$

PART II **Justify** your answers.

9. Consider the autonomous differential equation $y' = f(y)$, with a critical point c. **(8 marks)**

(a) Assume that $f'(c) > 0$. Graph $z = f(y)$ for values of y near c.



(b) Is c stable or unstable? Justify your answer.

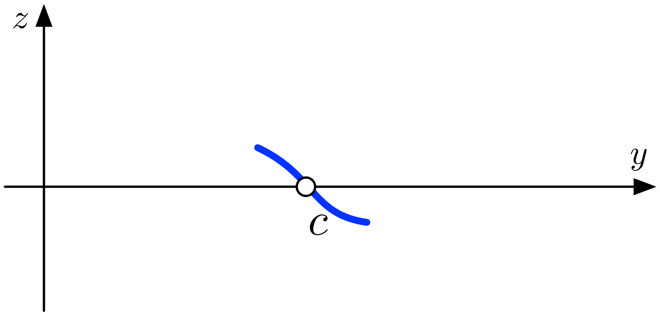
Solution. The critical point c is **unstable**, because in the graph from part **(a)**, we have the following

- If $y > c$, then $y' > 0$. This implies that y is becoming larger: moving away from c
- If $y < c$, then $y' < 0$. This implies that y is becoming smaller: moving away from c

So solutions with initial condition $y(t_0) = y_0$ for y_0 near c, will move away from c. Thus the critical point is unstable.

□

(c) Assume that $f'(c) < 0$. Graph $z = f(y)$ for values of y near c.



(d) Is c stable or unstable? Justify your answer.

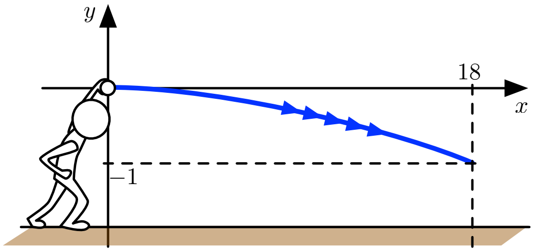
Solution. The critical point c is **stable**, because in the graph from part **(a)**, we have the following

- If $y > c$, then $y' < 0$. This implies that y is becoming smaller: moving towards c
- If $y < c$, then $y' > 0$. This implies that y is becoming larger: moving towards c

So solutions with initial condition $y(t_0) = y_0$ for y_0 near c, will converge to c. Thus the critical point is stable.

$\square$

10. You are a baseball pitcher and you want to throw a ball from your position to the catcher 18m away and 1m below your throwing position. Consider gravity only. **(10 marks)**



(a) If the pitcher throws the ball horizontally, how fast should he throw it? And how much time will it take for the ball to reach the catcher?

Proof. Solution.] Using Newton's 2nd Law of motion, we have

$$\vec{F} = m\vec{a}.$$

Define $\big(x(t), y(t)\big)$ as the position of the ball at time t. Then

$$\vec{a} = \big(x''(t), y''(t)\big) \qquad \text{and} \qquad \vec{F} = (0, -mg).$$

So we have the differential equations:

$$x''(t) = 0 \qquad \text{and} \qquad y''(t) = -g. \tag{$\star$}$$

The solution is

$$x(t) = u_0 t + x_0 \qquad \text{and} \qquad y(t) = -\frac{g}{2}t^2 + v_0 t + y_0,$$

assuming the initial conditions

$$\big(x(0), y(0)\big) = (0,0) \qquad \text{and} \qquad \big(x'(0), y'(0)\big) = (u_0, v_0).$$

The ball is thrown horizontally, so $v_0 = 0$.

Also, we want the ball to reach the catcher, so the solution must satisfy

$$x(T) = 18 \qquad \text{and} \qquad y(T) = -1.$$

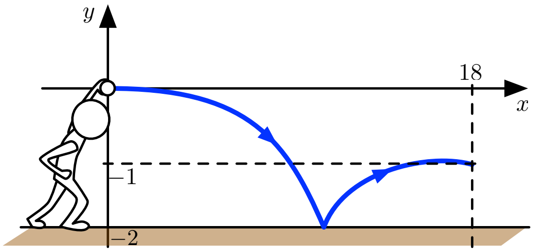
This implies that

$$\begin{cases} u_0 T = 18 \\ -\frac{g}{2}T^2 = -1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} u_0 = \frac{18}{T} \\ T = \sqrt{\frac{2}{g}} \end{cases} \quad \Leftrightarrow \quad \begin{cases} u_0 = 18\sqrt{\frac{g}{2}} \\ T = \sqrt{\frac{2}{g}} \end{cases}$$

The pitcher should throw the ball at $18\sqrt{\dfrac{g}{2}}$ m/s and it will take $\sqrt{\dfrac{2}{g}}$ s to reach the catcher.

□

(b) Assume that the pitcher is used to cricket: he throws the ball horizontally, the ball bounces once on the ground (2m below the throwing position), but loses a quarter of its velocity on the bounce.
With exactly one bounce, how fast should he throw the ball?



Solution. We need to split the calculations in two parts: before and after the bounce.

Before the bounce. We have

$$x(t) = u_0\, t \quad \text{and} \quad y(t) = -\frac{g}{2}t^2.$$

Define t_b = time when the ball touches the ground. Then $y(t_b) = -2$, which implies that

$$t_b^2 = \frac{4}{g} \quad \Leftrightarrow \quad t_b = \frac{2}{\sqrt{g}}.$$

This means that $x(t_b) = \dfrac{2u_0}{\sqrt{g}}$ and the velocity at the time of the bounce is

$$x'(t_b) = u_0 \quad \text{and} \quad y'(t_b) = -gt_b = -2\sqrt{g}.$$

After the bounce. The differential equation after the bounce is the same as before, so its solution is the same

$$x(t) = u_b(t - t_b) + x(t_b) \qquad \text{and} \qquad y(t) = -\frac{g}{2}(t-t_b)^2 + v_b(t-t_b) + y(t_b),$$

where

$$u_b = \frac{3}{4}u_0 \quad \text{and} \quad v_b = \frac{3}{4}2\sqrt{g} = \frac{3}{2}\sqrt{g}.$$

We have

$$x(t) = \frac{3}{4}u_0(t-t_b) + \frac{2u_0}{\sqrt{g}} \qquad \text{and} \qquad y(t) = -\frac{g}{2}(t-t_b)^2 + \frac{3}{2}\sqrt{g}(t-t_b) - 2.$$

We now need to find T such that

$$x(T) = 18 \quad \text{and} \quad y(T) = -1,$$

which implies

$$\begin{aligned} -\frac{g}{2}(T-t_b)^2 + \frac{3}{2}\sqrt{g}(T-t_b) - 2 = -1 \\ \frac{g}{2}(T-t_b)^2 - \frac{3}{2}\sqrt{g}(T-t_b) + 1 = 0 \\ T - t_b = \frac{\frac{3}{2}\sqrt{g} \pm \sqrt{\frac{9}{4}g - 2g}}{g} \\ T - t_b = \frac{3 \pm 1}{2\sqrt{g}} \end{aligned}$$

So we have two solutions

$$T = t_b + \frac{2}{\sqrt{g}} \quad \text{or} \quad T = t_b + \frac{1}{\sqrt{g}}.$$

The initial speed of the ball u_0 which is the solution of

$$u_0\left(\frac{3}{4}(T-t_b) + \frac{2}{\sqrt{g}}\right) = 18 u_0 = \frac{18\sqrt{g}}{\frac{3}{2}+2} = \frac{36}{7}\sqrt{g} \quad \text{m/s}.$$

or

$$u_0 = \frac{18\sqrt{g}}{\frac{3}{4}+2} = \frac{72}{11}\sqrt{g} \quad \text{m/s}.$$

□

11. **(a)** Find the general solution of the differential equation **(8 marks)**

$$\left(1-\cos(y)x^3\right)y'(x) = 3x^2\sin(y)+\cos(x).$$

(**Hint.** You can leave the solution in implicit form)

Solution. This equation is exact:

$$\underbrace{-\left(3x^2\sin(y)+\cos(x)\right)}_{M(x,y)}+\underbrace{\left(1-\cos(y)x^3\right)}_{N(x,y)}y'(x)=0,$$

and

$$M_y = -3x^2\cos(y) = N_x.$$

This means that we can solve it by finding $\psi(x,y)$ such that

$$\psi_x = M \quad \Leftrightarrow \quad \psi = \int M(x,y)\,dx = -x^3\sin(y)+\sin(x)+h(y).$$

We now find $h(y)$ using $\psi_y = N$:

$$\psi_y = -x^3\cos(y)+h'(y) = 1-x^3\cos(y) = N$$

so

$$h'(y)=1 \quad \Leftrightarrow \quad h(y) = y+C.$$

We can then take

$$\psi(x,y) = -x^3\sin(y)+\sin(x)+y,$$

and the general solution is given by

$$-x^3\sin(y)+\sin(x)+y = C.$$

□

(b) The differential equation

$$\left(\frac{1}{x} - \cos(y)x^2\right) y'(x) = 3x\sin(y) \qquad \text{is not exact.}$$

Find an integrating factor $\mu(x, y)$ to make this equation exact. Justify your answer.

Solution #1. This differential equation is very similar to the one is part **(a)**. If we multiply it by $\mu(x, y) = x$, we get

$$\underbrace{\left(1 - \cos(y)x^3\right)}_{N(x,y)} y'(x) = \underbrace{3x^2\sin(y)}_{-M(x,y)},$$

which is exact:

$$M_y = -3x^2\cos(y) = N_x.$$

☐

Solution #2. If we multiply the DE by $x\mu(x, y)$, we get

$$\underbrace{-3x\sin(y)\mu}_{\bar{M}} + \underbrace{\left(\frac{1}{x} - \cos(y)x^2\right)\mu}_{\bar{N}} y'(x) = 0,$$

Then

$$\begin{aligned} M_y &= -3x\cos(y)\mu - 3x\sin(y)\mu_y \\ N_x &= -\frac{1}{x^2}\mu + \frac{1}{x}\mu_x - 2x\cos(y)\mu - \cos(y)x^2\mu_x, \end{aligned}$$

so we want to choose μ such that

$$-3x\cos(y)\mu - 3x\sin(y)\mu_y = -\frac{1}{x^2}\mu + \frac{1}{x}\mu_x - 2x\cos(y)\mu - \cos(y)x^2\mu_x$$

If we choose $\mu = \mu(x)$, then we get

$$\begin{aligned} -3x\cos(y)\mu &= -\frac{1}{x^2}\mu + \frac{1}{x}\mu_x - 2x\cos(y)\mu - \cos(y)x^2\mu_x \\ x\cos(y)\left(x\mu_x - \mu\right) &= \frac{1}{x^2}\left(x\mu_x - \mu\right) \\ \left(x\cos(y) - \frac{1}{x^2}\right)\left(x\mu_x - \mu\right) &= 0 \end{aligned}$$

So we can choose $\mu(x)$ which satisfies

$$\begin{aligned} x\mu_x - \mu &= 0 \\ \frac{1}{\mu}\mu_x &= \frac{1}{x} \\ \mu &= x \end{aligned}$$

The integrating factor is $\mu(x) = x$. ☐

12. Consider the following initial value problem: **(8 marks)**

$$\begin{cases} 2y' = y^2 + y \\ y(0) = 1 \end{cases}$$

(a) Using Euler's Method with $h = \frac{1}{2}$, approximate the solution at $t = 1$.

Solution. First we write the differential equation as

$$y' = \frac{y(y+1)}{2}.$$

Euler's method yields:

$$\begin{aligned} y_0 &= 1 \\ y_1 &= 1 + \frac{1}{2} f(0,1) = 1 + \frac{1}{2} \cdot \frac{1 \cdot 2}{2} = \frac{3}{2} \\ y_2 &= \frac{3}{2} + \frac{1}{2} f\left(\frac{1}{2}, \frac{3}{2}\right) = \frac{3}{2} + \frac{1}{2} \cdot \frac{\frac{3}{2} \cdot \frac{5}{2}}{2} = \frac{3}{2} + \frac{15}{16} = \frac{39}{16} \end{aligned}$$

☐

(b) Find the solution of the initial value problem and compute the error of the approximation in **(a)** at $t = 1$.

Solution. The DE is separable:

$$\frac{y'}{y(y+1)} = \frac{1}{2}.$$

The solution satisfies

$$\int \frac{1}{y(y+1)}\, dy = \int \frac{1}{2}\, dt.$$

Using partial fractions, we write

$$\begin{aligned} \int \left(\frac{1}{y} - \frac{1}{y+1}\right) dy &= \frac{t}{2} + k \\ \ln|y| - \ln|y+1| &= \frac{t}{2} + k \\ \left|\frac{y}{y+1}\right| &= ce^{\frac{t}{2}} \\ \frac{y}{y+1} &= ce^{\frac{t}{2}} \end{aligned}$$

We can find c using the initial condition:

$$\frac{1}{2} = c.$$

So we can find y explicitly:

$$\begin{aligned} \frac{y}{y+1} &= \frac{1}{2}e^{\frac{t}{2}} \\ 2y &= (y+1)e^{\frac{t}{2}} \\ \left(2 - e^{\frac{t}{2}}\right) &= e^{\frac{t}{2}} \\ y &= \frac{e^{\frac{t}{2}}}{2 - e^{\frac{t}{2}}}. \end{aligned}$$

This gives

$$y(1) = \frac{e^{\frac{1}{2}}}{2 - e^{\frac{1}{2}}}.$$

So the error of the previous approximation is

$$\text{error} = |y(1) - y_2| = \left| \frac{e^{\frac{1}{2}}}{2 - e^{\frac{1}{2}}} - \frac{39}{16} \right|.$$

$\square$

(c) If we need to obtain an error 50 times smaller, which step size h should we choose?

Solution. Since Euler's method has an error of the order of h, to obtain an error 50 times smaller, we need h to be 50 times smaller. So we need

$$h = \frac{1}{2}\frac{1}{50} = \frac{1}{100}.$$

$\square$

13. Consider functions $p(t)$ and $g(t)$ continuous for $t \in (a,b)$ and consider the initial value problem **(8 marks)**

$$\begin{cases} y' + p(t)y = g(t) \qquad \text{for } t \in (a,b) \\ y(t_0) = y_0, \end{cases}$$

where $a < t_0 < b$. Let $\phi(t)$ and $\psi(t)$ be two solutions of this initial value problem. Show that $\phi(t) = \psi(t)$ for $t \in (a,b)$.

Hint. Split the proof in three steps:

(a) Define $F(t) = \phi(t) - \psi(t)$. Show that $F(t)$ is a solution of the initial value problem

$$\begin{cases} F' + p(t)F = 0 \qquad \text{for } t \in (a,b) \\ F(t_0) = 0. \end{cases}$$

(b) Solve this differential equation and find $F(t)$.

(c) Conclusion.

Solution. **(a)** First define $F(t) = \phi(t) - \psi(t)$. Then

$$\begin{aligned} F' + p(t)F &= \phi'(t) - \psi'(t) + p(t)\big(\phi(t) - \psi(t)\big) \\ &= \phi'(t) + p(t)\phi(t) - \big(\psi'(t) + p(t)\psi(t)\big) \\ &= g(t) - g(t) = 0. \end{aligned}$$

and

$$F(t_0) = \phi(t_0) - \psi(t_0) = y_0 - y_0 = 0.$$

(b) The equation is separable: we can write it as

$$\begin{aligned} \int \frac{1}{F}\, dF &= -\int p(t)\, dt \\ \ln|F| &= -\int p(t)\, dt + k \\ |F| &= c\, e^{-\int p(t)\, dt} \end{aligned}$$

We can use the initial condition to find $c = 0$ and we obtain

$$F(t) = 0.$$

(c) This implies that $\phi(t) = \psi(t)$.

□