MAT292 - Calculus III - Fall 2015

Term Test 2 - November 12, 2015

Time allotted: 90 minutes. Aids permitted: None.

Solution

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Last First

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection, turn off all cellular phones, and read all the instructions carefully.
- DO NOT start the test until instructed to do so.
- This test contains 14 pages (including this title page). Make sure you have all of them.
- You can use page 14 for rough work or to complete a question (**Mark clearly**).

  DO NOT DETACH PAGE 14.

GOOD LUCK!

PART I

No explanation is necessary.

For questions **1.** to **3.** , consider the system

$$\vec{x}\,' = A\vec{x}.$$

1. **(2 marks)** If A has eigenvalues $r = \alpha \pm \beta i$ and the equilibrium $\vec{0}$ is unstable, then

$$\alpha \in \big(\underline{\;0\;} \, , \, \underline{\;\infty\;} \big)$$

$$\beta \in \big(\underline{\;-\infty\;} \, , \, \underline{\;\infty\;} \big)$$

2. **(2 marks)** If A has eigenvalues $r_1 < 0 < r_2$ with eigenvectors $\vec{\xi}_1, \vec{\xi}_2$, then the equilibrium $\vec{0}$ is unstable. However, some solutions don't diverge to infinity.

Find all the possible initial conditions $\vec{x}(0) = \vec{x}_0$ such that $\lim_{t\to\infty} |\vec{x}(t)| = 0$:

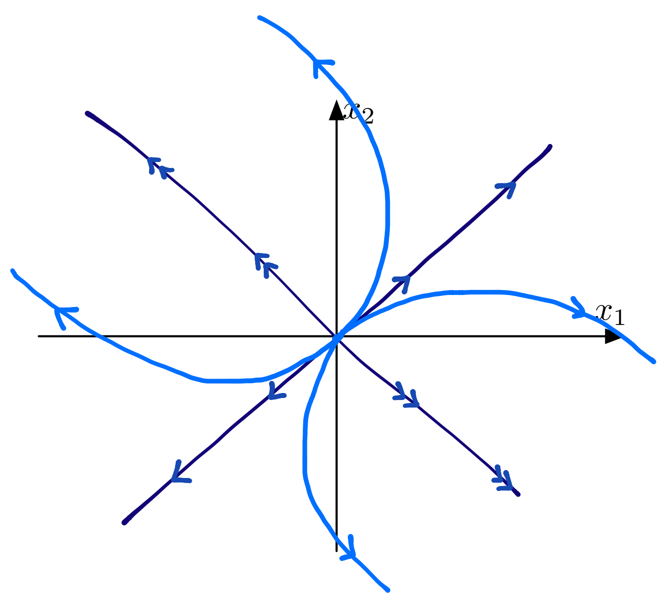
$\vec{x}_0 =$ $c\,\vec{\xi}_1$ for any constant $c \in \mathbb{R}$.

3. **(3 marks)** If A has the eigenvalues and eigenvectors

$$r_1 = 1 \quad , \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$r_2 = 5 \quad , \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

sketch the phase portrait.



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For questions **4.** to **5.** , consider the system

$$\vec{x}\,' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

4. **(2 marks)** The equilibrium solution is

$$\vec{x}_{\text{eq}} = \begin{pmatrix} \underline{\quad -1 \quad} \\ \underline{\quad 3 \quad} \end{pmatrix}.$$

5. **(2 marks)** The deviation from equilibrium $\vec{x}_h = \vec{x} - \vec{x}_{\text{eq}}$ satisfies the system

$$\vec{x}_h{}' = \begin{pmatrix} \underline{\quad 1 \quad} & \underline{\quad 1 \quad} \\ \underline{\quad -1 \quad} & \underline{\quad 1 \quad} \end{pmatrix} \vec{x}_h + \begin{pmatrix} \underline{\quad 0 \quad} \\ \underline{\quad 0 \quad} \end{pmatrix}.$$

6. **(3 marks)** Write a second-order linear differential equation with solutions

$$y_1 = e^{2t} \qquad \text{and} \qquad y_2 = -e^{2t} + e^{3t}.$$

$$y'' \quad + \quad \underline{\quad (-5) \quad} \quad y' \quad + \quad \underline{\quad 6 \quad} \quad y \quad = \quad \underline{\quad 0 \quad}$$

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PART II **Justify** your answers.

7. Consider the system of differential equations **(10 marks)**

$$\vec{x}\,' = \begin{pmatrix} 2a & -a \\ 5a & 0 \end{pmatrix} \vec{x},$$

where $a \neq 0$.

(a) **(8 marks)** Sketch all the possible phase portraits for this system. Justify your answer.

Eigenvalues. $\begin{vmatrix} 2a-\lambda & -a \\ 5a & -\lambda \end{vmatrix} = 0 \iff \lambda(\lambda - 2a) + 5a^2 = 0$

$\iff \lambda^2 - 2a\lambda + 5a^2 = 0$

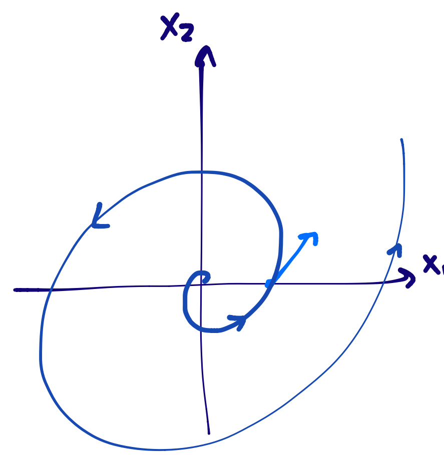
$\iff \lambda = a \pm \sqrt{a^2 - 5a^2} = a \pm 2ai$

$\Rightarrow$ Eigenvalues are always complex.

- If $\alpha > 0$, then we get a spiral source (unstable)

and $\vec{x}\,' = \begin{pmatrix} 2a & -a \\ 5a & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a \\ 5a \end{pmatrix}$ $\oplus$ $\oplus$

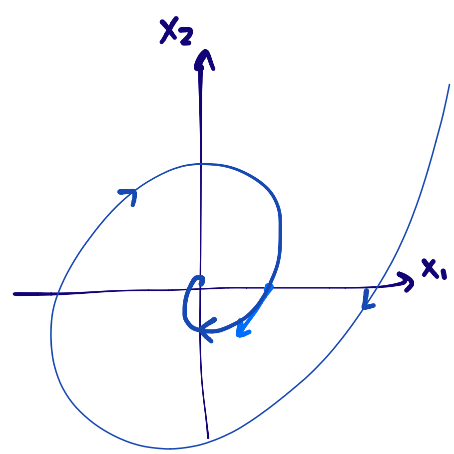
Counter-clockwise



- If $\alpha < 0$, then we get a spiral sink (stable)

and $\vec{x}\,' = \begin{pmatrix} 2a & -a \\ 5a & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2a \\ 5a \end{pmatrix}$ $\ominus$ $\ominus$

Clockwise



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(b) **(2 marks)** Give an example of a system of differential equations where the phase portrait is a centre.

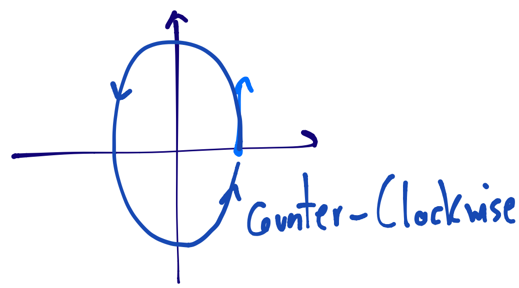
(1 extra mark) if it is clockwise.

The system $\vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$

has the eigenvalues : $\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 1 = 0 \Leftrightarrow \lambda = \pm i$

which is a <u>centre</u>.

Moreover $\vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \Rightarrow$



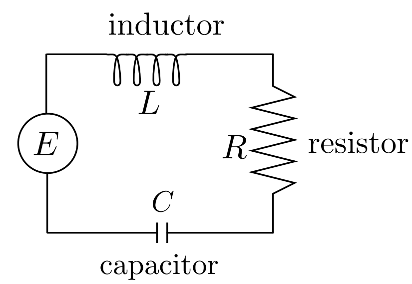
To get a clockwise centre, we need the system

$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}$

which has the same eigenvalues (so it is still a centre), but has the reverse orientation.

8. Consider an oscillator. We can model the charge $q(t)$ of the capacitor by using Kirchhoff's Second Law: **(6 marks)**

Sum of the voltage drops over the components of the circuit is equal to the impressed voltage.



We obtain the following DE:

$$Lq'' + Rq' + \frac{1}{C}q = E(t).$$

and $q' = i =$ current.

(a) **(2 marks)** Assuming $E(t) = 0$, find a condition on the constants R, L, C that makes sure that there will be oscillations in $q(t)$, and if the condition is not satisfied, there won't be any oscillations.

There will be oscillations if and only if the roots of the characteristic equation are complex (and not real).
The Char. eq. is $Lr^2 + Rr + \frac{1}{C} = 0 \Leftrightarrow r = \frac{-R \pm \sqrt{R^2 - 4\frac{L}{C}}}{2L}$ which are complex if and only if $R^2 - 4\frac{L}{C} < 0 \Leftrightarrow \boxed{R^2C - 4L < 0}$

(b) **(2 marks)** Still with $E(t) = 0$, if there are no oscillations, then what is the limiting behaviour of the solution? Justify using the Differential Equation.

If there are no oscillations, then $R^2 - 4\frac{L}{C} > 0$ and

$R^2 - 4\frac{L}{C} < R^2 \Rightarrow \sqrt{R^2 - 4L/C} < R \Rightarrow -R \pm \sqrt{R^2 - 4\frac{L}{C}} < 0$

This means that the solution will contain only terms of the form e^{rt} with $r<0$, so

$\lim_{t\to\infty} q(t) = 0$

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(c) **(2 marks)** Give an example of constants R, L, C and $E(t)$ such that $E(t)$ is bounded and

$$\lim_{t\to\infty} |q(t)| = \infty.$$

The complementary solution is always bounded:

- if r are complex, from (a), the real part is $-\frac{R}{2L} \leq 0$,
  so $q_c \cancel{\xrightarrow{t\to\infty} \infty}$ (if $R<0$, $q_c \xrightarrow{t\to\infty} 0$, and if $R=0$, then q_c will oscillate)

- otherwise, from (b), $q_c \xrightarrow{t\to\infty} 0$.

This means that the only way to make $|q| \longrightarrow \infty$ is through the particular solution.
We need to create resonance so that $E(t)$ can be bounded.
For that, we need an oscillating q_c:

$$\begin{cases} R=0 \\ R^2 - 4L/C < 0 \end{cases} \Rightarrow \begin{cases} R=0 \\ L=C=1 \end{cases} \text{(this one ex)}$$

Then $r = \pm i \Rightarrow \boxed{q_c = c_1 \cos(t) + c_2 \sin(t)}$

Then to create resonance, we need $E(t)$ to oscillate with the same frequency: $\boxed{E(t) = \cos(t)}$

Then the particular solution will have the form: $q_p = At\cos(t) + Bt\sin(t)$

Hence $q = \underbrace{c_1\cos(t) + c_2\sin(t)}_{\text{bounded}} + \underbrace{At\cos(t) + Bt\sin(t)}_{\text{unbounded}}$

So $\lim_{t\to\infty} |q(t)| = \infty$.

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9. Consider the initial-value problem **(10 marks)**

$$\begin{cases} ty'' + (2+t)y' + y = 0 \\ y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = v_0 \end{cases}$$

(a) **(2 marks)** For which values of t_0, y_0, v_0 can we guarantee that there is a unique solution? What is the domain of the solution?

We write the DE as $\begin{cases} y'' + \left(\frac{2}{t}+1\right)y' + \frac{1}{t}y = 0 \\ y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = v_0 \end{cases}$

This is a linear DE, so there is a unique solution if $\boxed{t_0 \neq 0}$. (y_0 and v_0 can have any real value)

If $t_0 > 0$, then the solution will be defined for $t > 0$.

If $t_0 < 0$, then the solution will be defined for $t < 0$.

(b) **(2 marks)** Show that $y_1 = \frac{1}{t}$ is a solution of the differential equation.

$y_1' = -\frac{1}{t^2}$ and $y_1'' = \frac{2}{t^3}$.

Then $ty_1'' + (2+t)y_1' + y_1 = \frac{2}{t^2} - (2+t)\frac{1}{t^2} + \frac{1}{t} = 0$ ✓

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(c) **(3 marks)** Find a second solution y_2 of the differential equation.

Using reduction of order, let $y_2 = \frac{u}{t}$. Then $y_2' = \frac{u'}{t} - \frac{u}{t^2}$, and

$y_2'' = \frac{u''}{t} - 2\frac{u'}{t^2} + 2\frac{u}{t^3}$, so

$$t y_2'' + (2+t) y_2' + y_2 = 0 \iff u\underbrace{\left(\frac{2}{t^2} - (2+t)\frac{1}{t^2} + \frac{1}{t}\right)}_{=0} + u'\underbrace{\left(\frac{2+t}{t} - \frac{2t}{t^2}\right)}_{=1} + u'' = 0$$

$$\iff u' + u'' = 0 \iff u + u' = c_1 \iff \frac{u'}{c_1 - u} = 1 \iff -\ln|c_1 - u| = t + \overline{c_2}$$

$$\iff c_1 - u = -c_2 e^{-t} \iff \boxed{u = c_1 + c_2 e^{-t}}$$

This gives $y_2 = \frac{u}{t} = \frac{c_1}{t} + c_2 \boxed{\frac{e^{-t}}{t}}$

We can "choose" a second solution $\boxed{y_2 = \frac{e^{-t}}{t}}$.

(d) **(1 mark)** Compute the Wronskian of the solutions found in **(b)** and **(c)** and show that it is never 0, as long as the solutions are defined.

Hint. $W[y_1, y_2] = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$.

$$W[y_1, y_2] = \begin{vmatrix} \frac{1}{t} & \frac{e^{-t}}{t} \\ -\frac{1}{t^2} & -\left(\frac{1}{t} + \frac{1}{t^2}\right)e^{-t} \end{vmatrix} = -\left(\frac{1}{t^2} + \frac{1}{t^3}\right)e^{-t} + \frac{e^{-t}}{t^3} = \frac{e^{-t}}{t^2} \neq 0 \text{ for any } t \neq 0.$$

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(e) **(2 marks)** Find the general solution to the DE

$$ty'' + (2+t)y' + y = t^2$$

step 1. $y_c = \frac{c_1}{t} + c_2 \frac{e^{-t}}{t}$ found in (a)-(c).

step 2. $y_p = At^2 + Bt + C$ which doesn't appear in y_c.

Then $y_p' = 2At + B$ and $y_p'' = 2A$.

So $ty_p'' + (2+t)y_p' + y_p = t^2 \iff 2At + (2+t)(2At+B) + (At^2+Bt+C) = t^2$

$\iff 2At + 4At + 2B + 2At^2 + Bt + At^2 + Bt + C = t^2$

$\iff 3At^2 + (6A+2B)t + (2B+C) = t^2$

$\iff \begin{cases} 3A = 1 \\ 6A+2B = 0 \\ 2B+C = 0 \end{cases} \iff \begin{cases} A = \frac{1}{3} \\ B = -1 \\ C = 2 \end{cases}$

So $y_p = \frac{t^2}{3} - t + 2$

step 3. the general solution is

$$\boxed{y = \frac{c_1}{t} + c_2\frac{e^{-t}}{t} + \frac{t^2}{3} - t + 2}$$

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10. When a baseball is flying through the air, spin will affect its motion. **(10 marks)**

Consider a baseball with $m = \frac{1}{7}\ kg$ and assume that $g = 9.8 = \frac{49}{5}\ m/s^2$.

Consider also the following functions:

- $x(t) = x$-position of the baseball at time t
- $y(t) = y$-position of the baseball at time t
- $v_\theta(t)$ = counter-clockwise velocity of the baseball in radians per second (if the ball is rotating clockwise, v_θ is negative) at time t

Then there are three forces acting on the baseball:

- Gravity
- Air drag: for simplicity, assume it is proportional to each component's velocity (including the spinning velocity) with proportionality constant $\gamma = 1$
- Magnus Effect (due to spin): Counterclockwise spin creates vertical lift proportional to spinning velocity (with proportionality constant $k = 1$)

(a) **(3 marks)** Define

- $v_x(t)$ = horizontal velocity of the baseball at time t
- $v_y(t)$ = vertical velocity of the baseball at time t

Find a system of DEs that describes the motion of the ball.

Use Newton's 2nd law: $ma = F$.

$\boxed{x}$ $mV_x' =$ air drag $= -V_x$

$\boxed{y}$ $mV_y' = \underbrace{\text{air drag}}_{-V_y} + \underbrace{\text{gravity}}_{-mg} + \underbrace{\text{Magnus Effect}}_{V_\theta}$

$\boxed{\theta}$ $mV_\theta' =$ air drag $= -V_\theta$

$$\begin{cases} m\,v_x' = -V_x \\ m\,v_y' = -V_y - mg + V_\theta \\ m\,v_\theta' = -V_\theta \end{cases}$$

(b) **(3 marks)** Assuming that the baseball was thrown horizontally with speed 40 m/s and spin 10 rad/s, find the solution of the system found in **(a)** .

Initial Conditions : $\begin{cases} v_y(0) = 0 & \text{(horizontal velocity vector)} \\ v_x(0) = 40 & \text{(speed equals } v_x \text{, because } v_y = 0) \\ v_\theta(0) = 10 \end{cases}$

$\boxed{V_x}$ $\quad v_x' = -\frac{1}{m} v_x \Rightarrow v_x = 40 e^{-t/m}$

$\boxed{V_\theta}$ $\quad v_\theta' = -\frac{1}{m} v_\theta \Rightarrow v_\theta = 10 e^{-t/m}$

int. factor

$\boxed{V_y}$ $\quad v_y' + \frac{1}{m} v_y = -g + \frac{10}{m} e^{-t/m} \iff \left(e^{t/m} v_y\right)' = -g e^{t/m} + \frac{10}{m}$

$\iff e^{t/m} v_y = -mg e^{t/m} + \frac{10}{m} t + C \iff v_y = -mg + \left(\frac{10}{m} t + C\right) e^{-t/m}$

$$\begin{cases} v_x(t) = 40 e^{-t/m} \\ v_y(t) = -mg + \left(\frac{10}{m} t + mg\right) e^{-t/m} \\ v_\theta(t) = 10 e^{-t/m} \end{cases}$$

(c) **(2 marks)** Assume that the ball was thrown from a height of 2 m. Find the position of the ball at time t.

Initial Conditions: $\begin{cases} x(0) = 0 \\ y(0) = 2 \end{cases}$

$x(t) = \int v_x(t)\,dt = -40 m e^{-t/m} + A \Rightarrow x = 40m\left(1 - e^{-t/m}\right)$

$y(t) = \int v_y(t)\,dt = -mgt + m^2 g e^{-t/m} + \frac{10}{m} \underbrace{\int t e^{-t/m}\,dt}_{-mte^{-t/m} + m\int e^{-t/m}\,dt = -mte^{-t/m} - m^2 e^{-t/m}} + B$

$y(t) = -mgt + m^2 g e^{-t/m} - 10 t e^{-t/m} - 10 m e^{-t/m} + B$

So $y(t) = -mgt + \left(m^2 g - 10t - 10m\right) e^{-t/m} - m^2 g + 10m + 2$

$$\begin{cases} x(t) = 40m\left(1 - e^{-t/m}\right) \\ y(t) = -mgt + \left(m^2 g - 10t - 10m\right) e^{-t/m} - m^2 g + 10m + 2 \end{cases}$$

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(d) **(2 marks)** Terms of the form e^{-rt} become very small very quickly. Ignore those terms in your solution and estimate when the ball lands on the ground.

$y(T)=0$ "$\Leftrightarrow$" $-mgT - m^2 g + 10m + 2 = 0 \Leftrightarrow T = \frac{2}{mg} + \frac{10}{g} - m$

ignoring the terms $e^{-t/m}$

$\Leftrightarrow T = \frac{10}{7} + \frac{50}{49} - \frac{1}{7}$

$\Leftrightarrow \boxed{T = \frac{9}{7} + \frac{50}{49} \approx 2.5 \text{ s}}$

(bonus) **(1 mark)** Once the ball lands, will it roll on the ground?

$V_a(T) = 40e^{-7T} \approx 40e^{-7\cdot(2.5)} = 40e^{-17.5} \approx 0$

$V_\theta(T) = 10e^{-7T} \approx 10e^{-17.5} \approx 0$

So the ball will have almost no horizontal velocity and almost no spin when it lands on the ground.

Thus $\boxed{\text{it will not roll on the ground.}}$

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