Tutorial Problems #1

MAT 292 – Calculus III – Fall 2015

Solutions

Solve $y' = 1 + 2y$. We can solve this equation using two methods: Integrating Factor or Separable Equation Method.

Method for Separable Equations. The goal is to write the DE as $f(y)y' = g(t)$. So we have

$$\frac{1}{1+2y}y' = 1.$$

Then we integrate with respect to t:

$$\int \frac{y'(t)}{1+2y(t)}\,dx = \int 1\,dx = t + c.$$

On the left integral, use the substitution $y = y(t)$ (so $dy = y'(t)\,dt$) and we obtain

$$\int \frac{1}{1+2y}\,dy = t + c_1 \quad \Leftrightarrow \quad \frac{1}{2}\ln|1+2y| = t + c_1 \quad \Leftrightarrow \quad |1+2y| = c_2 e^{2t} \qquad (c_2 = e^{2c_1})$$

Because the right-hand side e^{2t} is never 0, then the solution $1 + 2y$ can never switch from positive to negative (it would have to skip 0). This means that

- it is always positive: $1 + 2y = c_2 e^{2t}$;
- or it is always negative: $-1 - 2y = c_2 e^{2t} \quad \Rightarrow \quad 1 + 2y = -c_2 e^{2t}$.

These two options can be summarized by writing

$$1 + 2y = c_3 e^{2t},$$

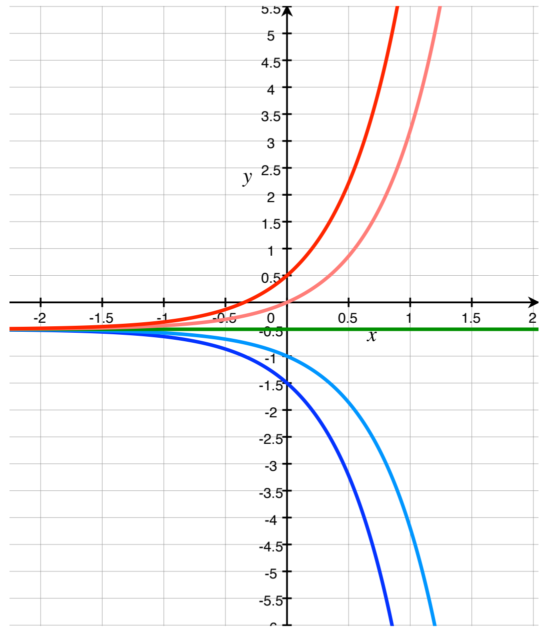
where $c_3 = \pm c_2$ (the sign depends on the initial condition).
We now solve for y:

$$y = -\frac{1}{2} + c_4 e^{2t},$$

where $c_4 = \frac{c_3}{2}$.

We can now study all the possible solutions for their behaviour:

- If $c_4 = 0 \Leftrightarrow y(0) = -\frac{1}{2}$, then the solution is constant: $y = -\frac{1}{2}$.
- If $c_4 > 0 \Leftrightarrow y(0) > -\frac{1}{2}$, then the solution will diverge to $+\infty$ exponentially.
- If $c_4 < 0 \Leftrightarrow y(0) < -\frac{1}{2}$, then the solution will diverge to $-\infty$ exponentially.



Solve $y' = -(1+2y)$. This problem is very similar to the previous one and the goal is to observe how the minus sign changes the behaviour of the solution rather than how to find the solution itself.

Method of Integrating Factor. The idea is to multiply the DE by a function $\mu(t)$ (integrating factor) so that the left-hand side of the DE looks like a product rule:
We get

$$\mu(t)y' + 2\mu(t)y = -\mu(t).$$

We want the left-hand side to be

$$\frac{d}{dt}\Big[\mu(t)y\Big] = \mu(t)y' + \mu'(t)y,$$

so we must choose $\mu(t)$ such that

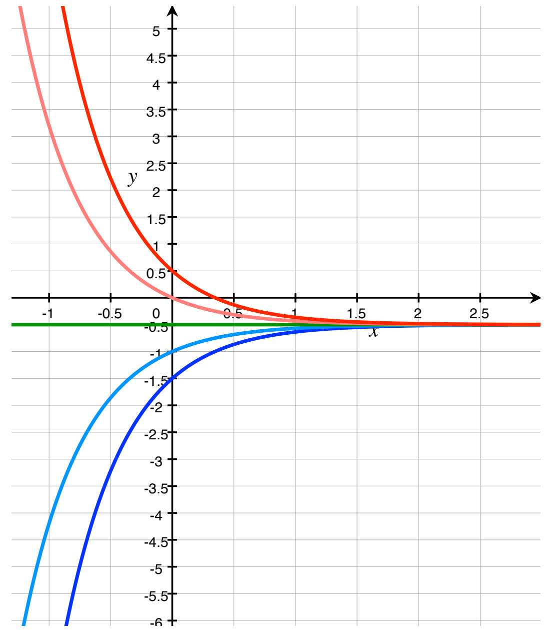
$$\mu'(t) = 2\mu(t).$$

Solve this DE to get $\mu(t) = e^{2t}$. Then the DE becomes

$$\frac{d}{dt}\Big[e^{2t}y\Big] = -e^{2t}.$$

Take an anti-derivative on both sides to get

$$e^{2t}y = -\frac{1}{2}e^{2t} + c \quad \Leftrightarrow \quad y = -\frac{1}{2} + ce^{-2t}.$$



We can now study all the possible solutions for their behaviour:

- If $c = 0 \Leftrightarrow y(0) = -\frac{1}{2}$, then the solution is constant: $y = -\frac{1}{2}$.
- If $c > 0 \Leftrightarrow y(0) > -\frac{1}{2}$, then the solution will converge to $-\frac{1}{2}$ exponentially.
- If $c < 0 \Leftrightarrow y(0) < -\frac{1}{2}$, then the solution will converge to $-\frac{1}{2}$ exponentially.

1.1.39.

(a) We need to find the differential equation that models this phenomenon.

First we need to define $y(t)$ = amount of drug present in the bloodstream (in mg). Then we know that

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}.$$

We now need to relate these two rates with $y(t)$:

- rate in $= 5 \cdot 100$ (mg/h)
- rate out $= 0.4y$

So the differential equation is

$$\frac{dy}{dt} = 500 - 0.4y.$$

(b) We now need to solve this differential equation. Using the method of integrating factor, we get

$$\frac{dy}{dt} + 0.4y = 500,$$

so we multiply the DE by $\mu(t) = e^{0.4t}$ to get

$$\frac{d}{dt}\left[e^{0.4t}y\right] = 500e^{0.4t}.$$

Take an antiderivative to obtain

$$e^{0.4t}y = \frac{500}{0.4}e^{0.4t} + c \quad \Leftrightarrow \quad y = 1250 + ce^{-0.4t}.$$

1.2.32. We need to find the general solution for this DE. For that we write

$$y' + \frac{t}{2}y = 1.$$

Using the integrating factor method, we multiply the DE by $\mu(t) = e^{t^2}$ to obtain

$$\frac{d}{dt}\left[e^{t^2}y\right] = e^{t^2},$$

so that

$$e^{t^2}y = \int_0^t e^{s^2}\,ds + c \quad \Leftrightarrow \quad \boxed{y = e^{-t^2}\int_0^t e^{s^2}\,ds + ce^{-t^2}}$$

using the formula (42) as hinted.
We now want to find the limit

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[e^{-t^2}\int_0^t e^{s^2}\,ds + ce^{-t^2}\right].$$

The last term of the limit is 0, independently of the value of c, so

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left[e^{-t^2}\int_0^t e^{s^2}\,ds\right].$$

To use L'Hôpital's rule, we need to write the function as a fraction:

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\frac{\displaystyle\int_0^t e^{s^2}\,ds}{e^{t^2}}.$$

Using L'Hôpital's Rule, we get

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\frac{e^{t^2}}{2te^{t^2}} = \lim_{t\to\infty}\frac{1}{2t} = 0.$$

We conclude that all solutions of this differential equation converge to 0 as $t \to \infty$.

1.1.29.

(a) To obtain the equation described, we need to use the quotient

$$u'(t_{j-1}) \cong \frac{u(t_j) - u(t_{j-1})}{\Delta t}.$$

We get

$$\begin{aligned}\frac{u(t_j) - u(t_{j-1})}{\Delta t} = -k\big(u(t_{j-1}) - T_0\big) \quad &\Leftrightarrow \quad u(t_j) - u(t_{j-1}) = -k\Delta t\big(u(t_{j-1}) - T_0\big)\\ &\Leftrightarrow \quad u(t_j) = u(t_{j-1}) - k\Delta t\big(u(t_{j-1}) - T_0\big)\\ &\Leftrightarrow \quad u(t_j) = (1 - k\Delta t)u(t_{j-1}) + k\Delta t T_0\end{aligned}$$

(b) Show by induction and then use the formula $\sum_{j=0}^{n-1} r^j = \frac{1-r^n}{1-r}$ for $|r| < 1$.

(c) Let $a_n = \left(1 - \frac{kt}{n}\right)^n$. Then

$$\ln a_n = n\ln\left(1 - \frac{kt}{n}\right).$$

We can then find

$$\begin{aligned}
\lim_{n\to\infty} \ln a_n &= \lim_{n\to\infty} n \ln\left(1 - \frac{kt}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1-\frac{kt}{n}\right)}{\frac{1}{n}} \\
&= \lim_{x\to\infty} \frac{\ln\left(1-\frac{kt}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{\frac{kt}{\cancel{x^2}}}{1-\frac{kt}{x}}}{-\frac{1}{\cancel{x^2}}} \\
&= \lim_{x\to\infty} -\frac{kt}{1-\frac{kt}{x}} = -kt
\end{aligned}$$

so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{\lim_{n\to\infty} \ln a_n} = e^{-kt}$$

Using the formula from (b), we easily obtain

$$\lim_{n\to\infty} u(t_n) = e^{-kt} u_0 + T_0 \left(1 - e^{-kt}\right).$$