Tutorial Problems #8

MAT 292 – Calculus III – Fall 2015

SOLUTIONS

4.1 - #19 A cubic block of side l and mass density ρ per unit volume is floating in a fluid of mass density ρ_0 per unit volume, where $\rho_0 > \rho$. If the block is slightly depressed and then released, it oscillates in the vertical direction. Assuming that the viscous damping of the fluid and air can be neglected, derive the differential equation of motion for this system. Hint: Use Archimedes's principle: An object that is completely or partially submerged in a fluid is acted on by an upward (buoyant) force equal to the weight of the displaced fluid.

Solution As stated in the question, we assume only vertical motion. Set $y = 0$ to be the surface of the water, so that $y = l/2$ is when the cube is completely removed from the water. Thus Newton's Second law states

$$F = ma \implies F_{gravity} + F_{buoyancy} = -mg + \underbrace{\rho_0 V(y)}_{=m_{water}} g = my''$$

The mass of the cube is given by $m = \rho V$, the volume of the cube is given by $V = l^3$, and the volume of the cube submerged is given by $V(y) = l^2(l/2 - y)$. Therefore we may write the differential as

$$\boxed{y'' + \frac{\rho_0 g}{\rho l} y = -g\left(1 - \frac{\rho_0}{2\rho}\right)}$$

□

4.2 - # 15 Can an equation $y'' + p(t)y' + q(t)y = 0$, with continuous coefficients, have $y = \sin(t^2)$ as a solution on an interval containing $t = 0$? Explain your answer.

Solution To check if this is possible, we assume p, q are continuous (since it's given), and check how the solution would solve the ODE (assuming it is a solution of course). We clearly have

$$y = \sin t^2, \quad \& \quad y' = 2t\cos t^2, \quad \& \quad y'' = 2\cos t^2 - 4t^2 \sin t^2$$

Plugging this back into the ODE gives the following:

$$2(1 + p(t)t)\cos(t^2) + (q(t) - 4t^2)\sin(t^2) = 0$$

Notice since the above should hold for t around 0, this implies the coefficients must be zero. i.e

$$1 + p(t)t = 0 \quad \& \quad 4t^2 + q(t) = 0$$

This fixes our choice of p and q, namely

$$p(t) = -\frac{1}{t} \quad \& \quad q(t) = 4t^2$$

This means $p(t)$ isn't continuous around $t = 0$, i.e. a contradiction to $p(t)$ being continuous. Therefore we cannot have $y = \sin(t^2)$ as a solution on an interval containing $t = 0$. □

4.2 - # 25 Prove Theorem 4.2.4 and Corollary 4.2.5

Theorem[4.2.4]: Let $K[x] = x' - P(t)x$, where the entries of P are continuous functions on an interval I. If x_1 and x_2 are continuously differentiable vector functions on I, and c_1 and c_2 are any constants, then,

$$K[c_1x_1 + c_2x_2] = c_1K[x_1] + c_2K[x_2]$$

Proof. By explicit computation we have

$$\begin{aligned} K[c_1x_1 + c_2x_2] =& (c_1x_1 + c_2x_2)' - P(t)(c_1x_1 + c_2x_2) \impliedby \text{Def'n of } K \\ =& c_1x_1' + c_2x_2' - c_1P(t)x_1 - c_2P(t)x_2 \impliedby x_1, x_2 \text{ are differentiable} \\ =& c_1(x_1' - P(t)x_1) + c_2(x_2' - P(t)x_2) \impliedby \text{rearranging} \\ =& c_1K[x_1] + c_2K[x_2] \impliedby \text{Def'n of } K \end{aligned}$$

□

Corollary[4.2.5]: Let $K[x] = x' - P(t)x$ and suppose the entries of P are continuous functions on an interval I. If x_1 and x_2 are two solutions of $K[x] = 0$, then the linear combination

$$x = c_1x_1(t) + c_2x_2(t)$$

is also a solution for any values of the constants c_1 and c_2.

Proof. Using the above theorem, we have

$$K[x] = K[c_1x_1 + c_2x_2] = c_1K[x_1] + c_2K[x_2]$$

Thus, if x_1 and x_2 are solutions, i.e. $K[x_1] = K[x_2] = 0$, we see

$$K[x] = 0$$

Therefore it is also a solution. □

4.2 - # 36 The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1 = \exp(-\delta x^2/2)$ is one solution and then find the general solution in the form of an integral.

Solution We begin by checking the solution.

$$y = e^{-\delta x^2/2}, \quad \& \quad y' = -\delta xy, \quad \& \quad y'' = -\delta y + \delta^2 x^2 y$$

Thus:

$$(-\delta y + \delta^2 x^2 y) + \delta(x(-\delta xy) + y) = 0$$

So it is indeed a solution. Recall that when you have the first solution, the full solution is readily found via

$$y(x) = y_1(x) \int \frac{W[y_1, y_2]}{y_1^2} dx$$

where the Wronskian is interpreted via Abel's formula:

$$W[y_1, y_2] = C \exp\left(-\int p(x)dx\right) = C \exp\left(-\delta \int xdx\right) = C \exp(-\delta x^2/2)$$

where $C \in \mathbb{R}$. Explicitly, we have y in the integral form of

$$\boxed{y(x) = Ce^{-\delta x^2/2} \int e^{\delta x^2/2} dx}$$

Since the inside is a Gaussian, we are unable to write a closed form. □

4.3 - #49 If the roots of the characteristic equation are real, show that a solution of $ay'' + by' + cy = 0$ can take on the value zero at most once.

Solution The assumption means the characteristic equation takes the form $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ with $\lambda_1, \lambda_2 \in \mathbb{R}$, thus

$$y(x) = \begin{cases} Ae^{\lambda_1 x} + Be^{\lambda_2 x}, & \lambda_1 \neq \lambda_2 \\ Ae^{\lambda_1 x} + Bxe^{\lambda_1 x}, & \lambda_1 = \lambda_2 \end{cases} \qquad A, B \in \mathbb{R}$$

In the non-repeated roots case we check how many solutions exist for $y(x) = 0$:

$$0 = Ae^{\lambda_1 x} + Be^{\lambda_2 x} \iff -1 = \frac{A}{B} e^{(\lambda_1 - \lambda_2)x} \iff x = \frac{\ln(-B/A)}{\lambda_1 - \lambda_2} \quad \text{if} \quad \frac{A}{B} < 0$$

otherwise there is no root. For the repeated root we see

$$0 = Ae^{\lambda_1 x} + Bxe^{\lambda_1 x} \iff 0 = A + Bx \iff x = -\frac{A}{B} \quad \text{if} \quad B \neq 0$$

otherwise there is no root. In both cases, we see there is at most one zero. □

4.3 - #44-46 Find a differential equation whose general solution is:

- $y = c_1 e^{2t} + c_2 e^{-3t}$
- $y = c_1 e^{-2t} + c_2 t e^{-2t}$
- $y = c_1 e^{-3t} \cos(4t) + c_2 e^{-3t} \sin(4t)$.

Solution Since we may read the eigenvalues directly from the solution, we know how the characteristic equation factors, thus we may find the differential equation using:

$$P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \implies y'' - (\lambda_1 + \lambda_2)y' + \lambda_1\lambda_2 y = 0$$

We see

- $y = c_1 e^{2t} + c_2 e^{-3t} \implies \lambda_1 = 2, \lambda_2 = -3 \implies \boxed{y'' + y' - 6y = 0}$
- $y = c_1 e^{-2t} + c_2 t e^{-2t} \implies \lambda_1 = -2, \lambda_2 = -2 \implies \boxed{y'' + 4y' + 4y = 0}$
- $y = c_1 e^{-3t}\cos(4t) + c_2 e^{-3t}\sin(4t) \implies \lambda_1 = -3 + 4i, \lambda_2 = -3 - 4i \implies \boxed{y'' + 6y' + 25y = 0}$

$\square$