Exam

MAT334 – Complex Variables – Spring 2016

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Solutions

Question 1

(a) Write the number z in standard form (i.e. $a + ib$ where $a, b \in \mathbb{R}$)

$$z = \frac{(1+i)^5}{7-2i}$$

(b) Determine the radius of convergence for the series

$$\sum_{n=1}^{\infty} \frac{n!}{(in)^n} z^n$$

Solution

(a) First note

$$(1+i)^5 = (\sqrt{2}e^{i\pi/4})^4(1+i) = (\sqrt{2})^4 e^{i\pi}(1+i) = -4-4i$$

Thus

$$z = \frac{(1+i)^5}{7-2i} = -4\frac{1+i}{7-2i} = -4\frac{1+i}{7-2i}\frac{7+2i}{7+2i} = -4\frac{5+9i}{53} = -\frac{20}{53} - i\frac{36}{53}$$

□

(b) By the ratio test with

$$a_n = \frac{n!}{(in)^n}$$

we see

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\frac{(in)^n}{(i(n+1))^{n+1}}\right| = \lim_{n\to\infty}\frac{n^n}{(n+1)^n} = \lim_{n\to\infty}\frac{1}{(1+1/n)^n} = \frac{1}{e}$$

Thus the radius of convergence is $1/e$.

□

Question 2 We consider the circle $\gamma = \{z = x+iy : x^2+y^2 = 9\}$ with the positive orientation and its interior D. Define

$$f : D \to \mathbb{C}, \quad f(z) = \oint_\gamma \frac{w^2-w+2}{w-z}dw$$

Compute $f(1)$ and $f'(1)$.

Solution Recall Cauchy's Integral Formula:

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z)} dw$$

Thus the function f may be written as

$$f(z) = 2\pi i(z^2 - z + 2)$$

Now we easily see that $f(1) = 4\pi i$ and

$$f'(z) = 2z - 1 \implies f'(1) = 2\pi i$$

□

Question 3 Determine the radius of convergence for the Taylor series of the analytic function

$$f(z) = \frac{z-1}{e^z - 1}$$

centered at $z_0 = i$

Solution Notice that $f(z)$ has a simple pole at $z = 0$. The Taylor series will not be defined there, thus the distance between the pole and z_0 will be the radius. i.e.

$$R = |z_{pole} - z_0| = |0 - i| = 1$$

□

Question 4 Compute the following integrals

(a)

$$\oint_{|z-\sqrt{3}-i|=1.5} \frac{\text{Log}(1+z^2)}{(z-\sqrt{3})^2} dz$$

(b)

$$\int_0^{2\pi} \frac{\cos(\theta)}{\cos(2\theta) + 2i} d\theta$$

(c)

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx$$

Solution

(a) The first integral is handled by Cauchy's Integral Formula, i.e.

$$f'(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(w)}{(w-z)^2} dw$$

In this case we see $f(z) = \text{Log}(1+z^2)$, so

$$f'(z) = \frac{2z}{1+z^2}$$

So

$$\oint_{|z-\sqrt{3}-i|=1.5} \frac{\text{Log}(1+z^2)}{(z-\sqrt{3})^2} dz = \sqrt{3}\pi i$$

(b) The second integral may be handled by symmetry, notice that

$$\cos(2\theta) = 2\cos^2\theta - 1$$

so the function is odd on the interval from $[0, 2\pi]$, so

$$\int_0^{2\pi} \frac{\cos(\theta)}{\cos(2\theta) + 2i} d\theta = 0$$

(c) The last integral we'll conquer with the Residue Theorem. Define the contour of the half disk of radius R with base on the real axis and setting

$$f(z) = \frac{z^2}{(z^2+1)^2}$$

and we'll integrate f around the contour. On the arc γ_R, we see that

$$\left| \int_{\gamma_R} f(z) dz \right| \leqslant \frac{const}{R}$$

Thus in the limit as $R \to \infty$ it doesn't contribute. We conclude

$$\lim_{R\to\infty} \int_\gamma f(z)dz = \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx = 2\pi i \times \text{Res}(f, i)$$

since the function has a pole or order 2 at i. We compute the residue using

$$\text{Res}(f, i) = \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z) = \lim_{z\to i} \frac{d}{dz} \frac{z^2}{(z+i)^2} = \lim_{z\to i} \frac{2z}{(z+i)^2} - 2\frac{z^2}{(z+i)^3} = -\frac{i}{2} + \frac{i}{4} = -\frac{i}{4}$$

Thus

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx = \frac{\pi}{2}$$

□

Question 5 Determine the maximum value of $|\text{Re}((1+i)z+1)|$ on the unit disc $|z| \leqslant 1$.

Solution Notice we may rewrite the function as

$$\text{Re}((1+i)z+1) = \text{Re}(x - y + 1 + i(x+y)) = x - y + 1$$

and by the maximum modulus principal, the maximum occurs on the boundary of the domain i.e. $|z| = 1$. In this case it's easy to see the max occurs at $(1/\sqrt{2}, -1/\sqrt{2})$ i.e.

$$\max_{z\in D} |\text{Re}((1+i)z+1)| = 1 + \sqrt{2}$$

Remark: You may also compute the critical points using the substitution $x = \cos\theta$ and $y = \sin\theta$, so we have

$$f(\theta) = \cos\theta - \sin\theta + 1$$

Then you may solve $f'(\theta) = 0$ to find the critical points of $\theta = -\pi/4$ and $3\pi/4$ to deduce that $\theta = -\pi/4$ is a maximum, and $\theta = 3\pi/4$ is a minimum.

□

Question 6 Determine the number of zeros of $f(z) = z^5 - 100z + 2$ in the disc $|z| < 10$.

Solution By Rouché's Theorem, we see if we set $g(z) = -z^5$, we have

$$|f + g| = |100z + 2| \leqslant 1002 < 100000 = 10^5 = |g|$$

on the boundary of the disc ($|z| = 10$). Thus f and g have the same number of zeros in the disc. g has a zero of order 5 at the origin, thus f has 5 zeros (counting multiplicities) in the disc.

□