Test 2

MAT334 – Complex Variables – Spring 2016

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$\boxed{\text{SOLUTIONS}}$

Question 1 Given $f : \mathbb{C} \to \mathbb{C}$, where $f(z) = (\bar{z} + i)|z|^2$.

- Compute $f'(0)$, by using the defintion
- Decide whether $f'(i)$ exists or it doesn't: if yes, compute it; if no, argue why

Solution By definition, we have that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

Thus we see that

$$f'(0) = \lim_{z \to 0} \frac{(\bar{z} + i)(z)(\bar{z})}{z} = \lim_{z \to 0} i\bar{z} + \bar{z}^2 = 0$$

The derivative at $f'(i)$ doesn't exist since $f(z)$ doesn't satisfy the Cauchy-Riemann Equations, i.e.

$$\frac{\partial f}{\partial \bar{z}}(i) \neq 0$$

$\square$

Question 2 Given the harmonic function

$$u(x, y) = e^x(x\cos(y) - y\sin(y)), \quad (x, y) \in \mathbb{R}^2$$

- Compute which of the following analytic functions has $u(x, y)$ as its real part:

$$a) ze^{i\left(z - \frac{\pi}{2}\right)}, \quad b) ize^{z - \frac{i\pi}{2}}, \quad c) ize^{iz}$$

- Use the Cauchy-Riemann equations to determine the harmonic conjugate v of u.

Solution Recall that

$$e^{iz} = \cos(z) + i\sin(z)$$

by Euler's identity. a) contains e^{-y} in the real component, so we may discard it. c) contains e^{-y} in the real component, so we discard it. For b) we see

$$ize^{z - \frac{i\pi}{2}} = -i^2 ze^z = ze^z = (x + iy)e^x e^{iy} = e^x(x\cos y - y\sin y) + ie^x(y\cos y + x\sin y)$$

Thus b) has the real part we want. Using the Cauchy-Riemann equations, we see

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \implies \begin{cases} v_y = e^x(x\cos y - y\sin(y)) + e^x \cos y \\ v_x = xe^x \sin(y) + e^x \sin(y) + e^x y\cos(y) \end{cases}$$

Integrating both equations gives us the imaginary part from the previous part:

$$\int v_y dy = e^x(y\cos y + x\sin y) + C \quad \& \quad \int v_x dx = e^x(y\cos y + x\sin y) + C$$

Thus $v(x,y) = e^x(y\cos y + x\sin y) + C$ as we originally saw. $\square$

Question 3 Determine the radius of convergence of the series

$$\sum_{k=1}^{\infty} \frac{(3k)!}{(k!)^3} z^{2k}$$

Solution Note that

$$a_{k+1} = \frac{(3k+3)(3k+2)(3k+1)}{(k+1)^3} a_k$$

Thus the ratio test shows that the series converges if

$$\lim_{k\to\infty} \left| \frac{a_{k+1} z^{2(k+1)}}{a_k z^{2k}} \right| = \lim_{k\to\infty} \frac{(3k+3)(3k+2)(3k+1)}{(k+1)^3} |z|^2 = 27|z|^2 < 1 \implies |z| < \frac{1}{\sqrt{27}}$$

thus the radius of convergence is $1/\sqrt{27}$. $\square$

Question 4 The power series expansion abou the origin of an analytic function $f : \mathbb{D} \to \mathbb{C}$, with $0 \in D$, has the form $f(z) = \sum_{n\geqslant 0} a_n z^n$.

- Determine a_0, a_1, a_2, a_3 for $f(z) = e^z(1 + z + z^2)$.
- Determine a_0, a_1, a_2, a_3 for

$$f(z) = \frac{e^z}{1 + z + z^2}$$

Solution Recall that

$$\exp(z) = \sum_{n\geqslant 0} \frac{z^n}{n!}$$

Thus the first series is given by

$$\begin{aligned} f(z) = e^z(1+z+z^2) = (1+z+z^2)\sum_{n\geqslant 0}\frac{z^n}{n!} &= (1+z+z^2) + z(1+z+z^2) + \frac{z^2(1+z+z^2)}{2} + \frac{z^3(1+z+z^2)}{6} + \mathcal{O}(z^4) \\ &= 1 + 2z + \frac{5}{2}z^2 + \frac{5}{3}z^2 + \mathcal{O}(z^4) \end{aligned}$$

The second series is found by

$$\frac{e^z}{1+z+z^2} = \sum_{n\geqslant 0} a_n z^n \implies \sum_{n\geqslant 0}\frac{z^n}{n!} = (1+z+z^2)\sum_{n\geqslant 0} a_n z^n = a_0 + (a_0+a_1)z + (a_0+a_1+a_2)z^2 + (a_0+a_1+a_2+a_3)z^3 + \mathcal{O}(z^4)$$

Comparing the first 4 terms of both series gives us 4 equations, and 4 unknowns:

$$\begin{cases} a_0 = 1 \\ a_0 + a_1 = 1 \\ a_0 + a_1 + a_2 = \frac{1}{2} \\ a_0 + a_1 + a_2 + a_3 = \frac{1}{6} \end{cases} \implies \begin{cases} a_0 = 1 \\ a_1 = 0 \\ a_2 = -\frac{1}{2} \\ a_3 = \frac{2}{3} \end{cases} \implies f(z) = \frac{e^z}{1+z+z^2} = 1 - \frac{z^2}{2} + \frac{2z^3}{3} + \mathcal{O}(z^4)$$

$\square$

Question 5 Determine the power series expansion about the origin of the function

$$\frac{1+z}{1+z^2}$$

Solution Note the geometric series is given by

$$\frac{1}{1-z} = \sum_{n\geqslant 0} z^n$$

Thus

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n\geqslant 0}(-z^2)^n = \sum_{n\geqslant 0}(-1)^n z^{2n}$$

then multiplying by $1+z$ gives

$$\frac{1+z}{1+z^2} = (1+z)\sum_{n\geqslant 0}(-1)^n z^{2n} = \sum_{n\geqslant 0} a_n z^n$$

where

$$a_n = \begin{cases} (-1)^{n/2} & \text{when } n \text{ is even} \\ (-1)^{(n-1)/2} & \text{when } n \text{ is odd} \end{cases}$$

$\square$

Question 6 Determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{(2+i)^n}$$

by summing-up an appropriate power series in the variable z and evaluating at an appropriate point.

Solution We know the geometric series is given by ($|z| < |a|$)

$$\frac{1}{a-z} = \frac{1}{a}\left(\frac{1}{1-z/a}\right) = \frac{1}{a}\sum_{n\geqslant 0}\left(\frac{z}{a}\right)^n$$

and the derivative of the series is given by

$$\frac{1}{(a-z)^2} = \frac{1}{a}\sum_{n\geqslant 1} n\frac{z^{n-1}}{a^n}$$

then multiplying by z and dividing the top and bottom by a^2, we see

$$\frac{\frac{z}{a}}{(1-\frac{z}{a})^2} = \sum_{n\geqslant 1} n\left(\frac{z}{a}\right)^n$$

Thus if we let

$$\frac{z}{a} = \frac{1}{2+i}$$

we see that

$$\sum_{n=1}^{\infty}\frac{n}{(2+i)^n} = \frac{1}{2+i}\frac{1}{(1-1/(2+i))^2} = \frac{2+i}{(1+i)^2} = \frac{1}{2} - i$$

$\square$