# Tutorial 1

MAT334 – Complex Variables – Spring 2016

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SOLUTIONS

**1.1 - #15** Show that the triangle with vertices at $0, z$, and $w$ is equilateral if and only if

$$|z|^2 = |w|^2 = 2\Re(z\bar{w})$$

**Solution** To check if the triangle is equilateral, we only need to check all sides are the same length. Thus we require

$$|z|^2 = |w|^2 = |z-w|^2$$

Expanding out the last term shows

$$|z-w|^2 = |z|^2 + |w|^2 - 2\Re(z\bar{w})$$

Therefore, we conclude

$$|z|^2 = |z-w|^2 = |z|^2 + |w|^2 - 2\Re(z\bar{w}) \implies |w|^2 = 2\Re(z\bar{w})$$

which shows the forward direction. The reverse direction is clear by the middle identity.

**1.1.1 - #8** Define the complex conjugate, $\bar{z}$, of $z = (x, y)$ by $\bar{z} = (x, -y)$. Show that $z\bar{z} = (|z|^2, 0)$

**Solution** We may write

$$z\bar{z} = (x+iy)(x-iy) = x^2 - ixy + ixy + y^2 = x^2 + y^2 = |z|^2 = (|z|^2, 0)$$

**1.1.1 - #10** Let $z = (x, y)$. Show that

$$|x| \leqslant |z|, \quad |y| \leqslant |z|, \quad |z| \leqslant |x| + |y|$$

**Solution** We see that

$$|z| = |x+iy| = \sqrt{z\bar{z}} = \sqrt{x^2+y^2} \geqslant \sqrt{x^2} = |x|$$

and by neglecting $x$ we obtain $|z| \leqslant |y|$. The last bound is found by considering

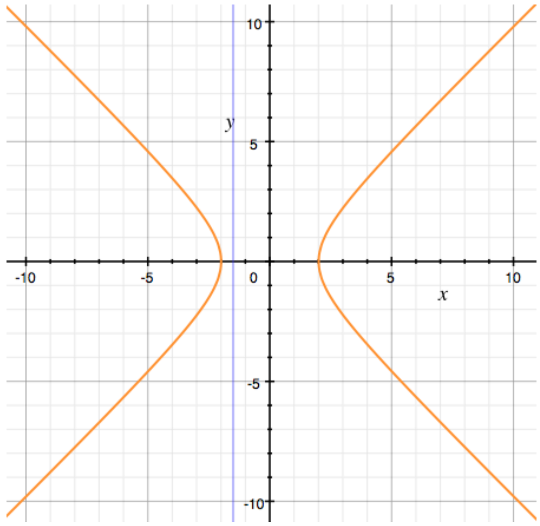
$$(|x| + |y|)^2 = x^2 + y^2 + 2|x||y| \geqslant x^2 + y^2 = |z|^2 \implies |x| + |y| \geqslant |z|$$

**1.2 - #7,8** Describe the locus of points $z$ satisfying $\Re(z^2) = 4$, then $|z-1|^2 = |z+1|^2 + 6$.

**Solution** It's easy to rewrite the restrictions in terms of $x$ and $y$. We see

$$\Re(z^2) = 4 \implies \Re(x^2 - y^2 + 2ixy) = \boxed{x^2 - y^2 = 4}$$

$$|z-1|^2 = |z+1|^2 + 6 \implies (x-1)^2 + y^2 = (x+1)^2 + y^2 + 6 \implies -2x = 2x + 6 \implies \boxed{x = -\frac{3}{2}}$$




**1.2 - #19** Let $p$ be a positive real number and let $\Gamma$ be the locus of points $z$ satisfying $|z-p| = cx$, $z = x+iy$. Show that $\Gamma$ is an ellipse if $c \in (0,1)$, a parabola if $c = 1$ and a hyperbola if $c \in (1,\infty)$.

**Solution** Expanding the restriction reveals

$$|z-p| = cx \implies \sqrt{(x-p)^2 + y^2} = cx \implies x^2 - 2xp + p^2 + y^2 = c^2x^2, \quad x > 0$$

$$\implies \boxed{(1-c^2)x^2 - 2xp + y^2 = 0, \quad x > 0}$$

We know the sign of the quadratic component is what determines the behaviour. Thus

$$c \in (0,1) \implies (1-c^2) > 0 \implies \frac{\tilde{x}^2}{a^2} + y^2 = 1 \implies \text{ellipse}$$

$$c = 0 \implies (1-c^2) = 0 \implies x = \frac{y^2}{2p} \implies \text{parabola}$$

$$c \in (1,\infty) \implies (1-c^2) < 0 \implies -\frac{\tilde{x}^2}{a^2} + y^2 = 1 \implies \text{hyperbola}$$

where $\tilde{x}$ is the appropriate translation of $x$.

**1.2 - #26** Find all solutions of

$$z^3 = 8$$

**Solution** Let $z = 2e^{i\theta}$, then we see

$$e^{3i\theta} = 1 \implies 3\theta = 2k\pi, \quad k \in 0, 1, 2$$

So the three roots to the equation are given by

$$z = 2, 2e^{i\frac{2}{3}\pi}, 2e^{i\frac{4}{3}\pi}$$

**1.2 - #29** Let $b$ and $c$ be complex numbers. Show that the roots of the quadratic equation $z^2 + bz + c = 0$ are complex conjugates of each other if and only if the quantity $b^2 - 4c$ is real and negative,$b$ is real, and $c$ is positive.

**Solution** By the quadratic formula, we see the roots are given by

$$z_{\pm} = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = \frac{-b \pm \sqrt{\Delta}}{2}$$

Thus the reverse direction ( $\Longleftarrow$ ) is obvious. The forward direction ( $\Longrightarrow$ ) follows If we require the roots to be complex conjugates, we see that this means

$$z_+ = \bar{z}_-$$

To check this is the case, we'll first check if $|z_+|^2 = z_+ z_- > 0$.

$$z_+ z_- = \frac{b^2 - (b^2 - 4c)}{4} = c \implies c > 0$$

Next we see that

$$\begin{aligned} z_+ = \bar{z}_- \implies -\bar{b} + \overline{\sqrt{\Delta}} = -b - \sqrt{\Delta} &\implies \overline{\sqrt{\Delta}} + \sqrt{\Delta} = \bar{b} - b \\ &\implies \overline{\Delta} + \Delta + 2|\sqrt{\Delta}| = \bar{b}^2 + b^2 - 2|b|^2 \\ &\implies -8c + 2|b^2 - 4c| = -2|b|^2 \\ &\implies |b^2 - 4c| = -(|b|^2 - 4c) \end{aligned}$$

Which shows that $b$ is real and $b^2 - 4c$ is negative (since $|x + y| = |x| + |y|$ only if $x = ay$ with $a \in \mathbb{R}$)