Tutorial 3

MAT334 – Complex Variables – Spring 2016

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SOLUTIONS

1.5 -# 3,8,11,13,14 Find the value(s) of

$$\log(1+i\sqrt{3}), \quad \exp(\mathrm{Log}(3+2i)), \quad i^{\sqrt{3}}, \quad \log((1-i)^4) \quad \& \quad \exp\left[\pi\left(\frac{i+1}{\sqrt{2}}\right)^4\right]$$

Solution

$$\log(1+i\sqrt{3}) = \log|1+i\sqrt{3}| + i\arg(1+i\sqrt{3}) = \log 4 + i\left(\frac{\pi}{3}+2k\pi\right), \quad k\in\mathbb{Z}$$

$$\exp(\mathrm{Log}(3+2i)) = 3+2i$$

$$i^{\sqrt{3}} = e^{\sqrt{3}(i\pi/2+2\pi ik)} = \exp i\left(\sqrt{3}\pi/2+2\sqrt{3}\pi k\right), \quad k\in\mathbb{Z}$$

$$\log((1-i)^4) = 4\log(1-i) = 4(\log|1-i|+i\arg(1-i)) = 4\log\sqrt{2}+4i\left(\frac{\pi}{4}+2\pi k\right) = 2\log 2+i\pi(1+8k), \quad k\in\mathbb{Z}$$

$$\exp\left[\pi\left(\frac{1+i}{\sqrt{2}}\right)^4\right] = \exp\left[\pi(e^{\frac{\pi i}{4}})^4\right] = e^{\pi e^{\pi i}} = e^{-\pi}$$

1.5 - # 17 Show that $\cos z = 0$ if and only if $z = \pi/2 + n\pi$, $n \in \mathbb{Z}$. Show that $\sin z = 0$ if and only if $z = n\pi$, $n \in \mathbb{Z}$. That is, extending $\sin z$ and $\cos z$ from the real axis to the whole plane does not introduce any new zeros.

Solution We rewrite cosine and sine into their exponential forms via Euler's Identity to expand to $\mathbb{C}$,

$$\cos z = \frac{1}{2}(e^{iz}+e^{-iz}) \quad \& \quad \sin z = \frac{1}{2i}(e^{iz}-e^{-iz})$$

Thus

$$\cos z = 0 \iff e^{iz} = -e^{-iz} = e^{-iz+\pi i} \quad \& \quad \sin(z) = 0 \iff e^{iz} = e^{-iz}$$

Suppose that $z = x+iy$.Then applying log to both sides shows

$$\cos(z) = 0 \iff -y+ix = y+i(-x+\pi+2n\pi) \iff \begin{cases} y=-y \\ x=-x+\pi+2n\pi \end{cases} \iff z = \frac{\pi}{2}+n\pi, \quad n\in\mathbb{Z}$$

$$\sin(z) = 0 \iff -y+ix = y+i(-x+2n\pi) \iff \begin{cases} y=-y \\ x=-x+2n\pi \end{cases} \iff z = n\pi, \quad n\in\mathbb{Z}$$

□

1.5 - # 21 Define

$$\cosh z = \frac{1}{2}(e^z + e^{-z}) \quad \& \quad \sinh z = \frac{1}{2}(e^z - e^{-z})$$

Show that the following identities hold:

$$\cosh^2(z) - \sinh^2(z) = 1$$

$$\cosh z = \cos iz$$

$$\sinh z = -\sin(iz)$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y$$

$$|\sinh z|^2 = \sinh^2 x + \sin^2 y$$

Solution The first follows immediately,

$$4(\cosh^2(z) - \sinh^2(z)) = (e^z + e^{-z})^2 - (e^z - e^{-z})^2 = 2 + 2 = 4$$

The second and third follow immediately by Euler's identity (see previous question). The forth and fifth follows by a short computation

$$|\cosh z|^2 = \frac{1}{4}(e^{x+iy} + e^{-x-iy})(e^{x-iy} + e^{-x+iy}) = \left(\frac{e^x - e^{-x}}{2}\right)^2 + \left(\frac{e^{iy} + e^{-iy}}{2}\right)^2 = \sinh^2 x + \cos^2 y$$

$$|\sinh z|^2 = \frac{1}{4}(e^{x+iy} - e^{-x-iy})(e^{x-iy} - e^{-x+iy}) = \left(\frac{e^x - e^{-x}}{2}\right)^2 + \left(\frac{e^{iy} - e^{-iy}}{2}\right)^2 = \sinh^2 x + \sin^2 y$$

□

1.5 - # 24 Let D be the domain obtained by deleting the ray $\{x : x \leqslant 0\}$ from the plane, and let $G(z)$ be a branch of $\log z$ on D. Show that G maps D onto a horizontal strip of width 2π

$$\{x + iy : -\infty < x < \infty, c_0 < y < c_0 + 2\pi\}$$

and that the mapping is one-to-one on D.

Solution This is easy to see, if we fix the branch the log maps to, we obtain a nice bijection:

$$G(z) = \log|z| + i(\text{Arg}(z) + c_0 + \pi)$$

where $c_0 \in \mathbb{R}$. Thus

$$\begin{aligned}\{z = Re^{i\theta} : R \in (0,\infty), \theta \in (-\pi,\pi)\} \to &\{w = \log R + i(\theta + c_0 + \pi) : R \in (0,\infty), \theta \in (-\pi,\pi)\} \\ = &\{w = x + iy : x \in \mathbb{R}, y \in (c_0, c_0 + 2\pi)\}\end{aligned}$$

It is 1-to-1 since

$$G(z) = G(w) \implies \log|z| + i(\text{Arg}(z) + c_0 + \pi) = \log|w| + i(\text{Arg}(w) + c_0 + \pi) \implies \begin{cases} \log|z| = \log|w| \\ \text{Arg}(z) = \text{Arg}(w) \end{cases} \implies z = w$$

□

1.5 - # 27 Let $0 < \alpha < 2$. Show that an appropriate choice of $\log z$ for $f(z) = z^\alpha = \exp[\alpha \log z]$ maps the domain $\{x + iy : y > 0\}$ both one-to-one and onto the domain $\{w : 0 < \arg w < \alpha\pi\}$. Show that f also carries the boundary to the boundary.

Solution Again, this is easier in Polar coordinates, since

$$f_k(z) = f_k(R, \theta) = z^\alpha = R^\alpha e^{i\alpha\theta + 2i\alpha\pi k} \quad k \in \mathbb{Z}$$

We see choosing the principal branch, i.e. $k = 0$, we obtain that

$$f_0(z) = R^\alpha e^{i\alpha\theta}$$

is a map that takes

$$\{x + iy : y > 0\} = \{Re^{i\theta} : \theta \in (0, \pi)\} \to \{w = R^\alpha e^{i\alpha\theta} : \theta \in (0, \pi)\} = \{w : 0 < \arg w < \alpha\pi\}$$

Clearly it takes the boundary to the boundary since

$$\{z : y = 0\} \to \{z : y = 0, x \geqslant 0\} \cup \{z = R^\alpha e^{i\alpha\pi} : R > 0\}$$

□

1.5 - # 29 Show directly that if ζ is any value of

$$-i \log(iz + \sqrt{1 - z^2})$$

then $\sin \zeta = z$. Likewise, show that if ξ is any value of

$$\frac{i}{2} \log \left(\frac{1 - iw}{1 + iw} \right)$$

then $\tan \xi = w$.

Solution Using our exponential form of sine, we see

$$\begin{aligned}
\sin \zeta =& \frac{1}{2i}(e^{\log(iz+\sqrt{1-z^2})} - e^{-\log(iz+\sqrt{1-z^2})}) \\
=& \frac{1}{2i} \left(iz + \sqrt{1 - z^2} - \frac{1}{iz + \sqrt{1 - z^2}} \right) \\
=& \frac{1}{2i} \left(\frac{(iz + \sqrt{1 - z^2})^2 - 1}{iz + \sqrt{1 - z^2}} \right) \\
=& \frac{1}{2i} \left(\frac{-z^2 + 1 - z^2 - 1 + 2iz\sqrt{1 - z^2}}{iz + \sqrt{1 - z^2}} \right) \\
=& z
\end{aligned}$$

Then using the exponential form of tangent, we see

$$\begin{aligned}
\tan\xi &= -i\frac{e^{2i\xi}-1}{e^{2i\xi}+1}\\
&= -i\frac{\exp\left(-\log\left(\frac{1-iw}{1+iw}\right)\right)-1}{\exp\left(-\log\left(\frac{1-iw}{1+iw}\right)\right)+1}\\
&= -i\frac{\frac{1+iw}{1-iw}-1}{\frac{1+iw}{1-iw}+1}\\
&= -i\frac{1+iw-1+iw}{1+iw+1-iw}\\
&= w
\end{aligned}$$

□