Assignment 7

MATC34 – Complex Variables – Fall 2015

Solutions

Question 1 Find the principal part of the Laurent Series and identify the residue for

$$f(z) = \frac{\sin z^2}{z^3(1+z)}$$

at $z = 0$.

Solution We know the power series for $\sin z$ and $1/(1-z)$ already, so we have for free that

$$\sin z^2 = z^2 - \frac{z^6}{3!} + \mathcal{O}(z^{10})$$

$$\frac{1}{1+z} = 1 - z + \mathcal{O}(z^2)$$

Thus by simple multiplication we see

$$\begin{aligned} f(z) =& \frac{\sin z^2}{z^3(1+z)} \\ =& \frac{1}{z^3}\left(z^2 - \frac{z^6}{3!} + \mathcal{O}(z^{10})\right)\left(1 - z + \mathcal{O}(z^2)\right) \\ =& \underbrace{\frac{1}{z}}_{\text{principal part}} \quad -1 + z + \mathcal{O}(z^2) \end{aligned}$$

Thus we can read off the principal part, and we see the coefficient on the $1/z$ term is 1, which is also the residue of $f(z)$ at $z_0 = 0$. $\square$

Question 2 Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left(\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right), \quad z \neq 1$$

Solution We first compute the power series of e^z entered around $z = -1$ using the trick from last week. We have

$$e^z = \frac{1}{e}e^{z+1} = \frac{1}{e}\sum_{n=0}^{\infty}\frac{(z+1)^n}{n!}$$

Thus

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\sum_{n=0}^{\infty}\frac{(z+1)^{n-2}}{n!} = \frac{1}{e}\left(\sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right), \quad z \neq 1$$

$\square$

Question 3 Evaluate

$$\int_{|z|=1} \cos\left(\frac{1}{z^2}\right) e^{1/z} dz$$

Solution We know the power series representation about $z_0 = 0$ for both functions, thus we see

$$\begin{aligned}\cos\left(\frac{1}{z^2}\right) e^{1/z} &= \left(1 - \frac{1}{2!z^4} + \mathcal{O}\left(\frac{1}{z^8}\right)\right)\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right) \\ &= 1 + \frac{1}{z} + \frac{1}{2z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\end{aligned}$$

As we already know, we have that

$$\int_{|z|=1} \frac{dz}{z^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1 \end{cases}$$

Thus we see

$$\int_{|z|=1} \cos\left(\frac{1}{z^2}\right) e^{1/z} dz = 2\pi i$$

□

Alternate Solution Take a change of variables to the integrand, $w = 1/z$, we see $dw = -dz/z^2$, therefore

$$\int_{|z|=1} \cos\left(\frac{1}{z^2}\right) e^{1/z} dz = \int_{|w|=1} \frac{\cos\left(w^2\right) e^w}{w^2} dw$$

note the double negative via the change of orientation. By expanding out both terms into power series, we see

$$\int_{|w|=1} \frac{\cos\left(w^2\right) e^w}{w^2} dw = \int_{|w|=1} \frac{1}{w^2}\left(1 - \frac{w^2}{2} + \mathcal{O}(w^4)\right)\left(1 + w + \frac{w^2}{2} + \mathcal{O}(w^3)\right) dw = \int_{|w|=1} \left(\frac{1}{w^2} + \frac{1}{w} + \mathcal{O}(w)\right) dw$$

Thus we see

$$\int_{|w|=1} \left(\frac{1}{w^2} + \frac{1}{w} + \mathcal{O}(w)\right) dw = 2\pi i$$

□

Question 4 Evaluate the integral using Cauchy's Residue theorem.

$$\int_C \frac{\exp(-z)}{\sin(z^2 + z)} dz$$

where $C = \{z : |z| = 3/2\}$.

Solution We know e^{-z} is an entire function, thus we have to find the zeros of $\sin(z^2 + z)$ (i.e. the poles in question) to apply the residue theorem. We see

$$\sin(z(z+1)) = 0 \implies z^2 + z = n\pi, \quad n \in \mathbb{Z} \implies z = \frac{-1 \pm \sqrt{1 + 4n\pi}}{2}$$

Since C is just the circle of radius 1 centred at 0, we see 2 poles are inside, specifically at $z = -1$ and $z = 0$. We calculate the residues:

$$\operatorname{Res}(f, 0) = \lim_{z\to 0} z\frac{\exp(-z)}{\sin(z^2 + z)} = \lim_{z\to 0} \frac{z}{\sin(z^2 + z)} = \lim_{z\to 0} \frac{1}{(2z + 1)\cos(z^2 + z)} = 1$$

$$\text{Res}(f,-1) = \lim_{z\to -1}(z+1)\frac{\exp(-z)}{\sin(z^2+z)} = \lim_{z\to -1}\frac{e(z+1)}{\sin(z^2+z)} = \lim_{z\to -1}\frac{e}{(2z+1)\cos(z^2+z)} = -e$$

The residue theorem now tells us

$$\int_C \frac{\exp(-z)}{\sin(z^2+z)}dz = 2\pi i\,(1-e)$$

□

Question 5 Calculated the integral

$$\int_C \frac{\cos(\pi z)}{\sin(\pi z)(1+z^4)}dz$$

where C is the counterclockwise oriented rectangle with vertices at $z_{1,2} = \frac{\pm 3+i}{2}$, $z_{3,4} = \frac{\pm 3-i}{2}$

Solution We see the poles of the function in question are given by

$$\sin(\pi z) = 0 \implies z = k \quad k\in\mathbb{Z} \quad \& \quad 1+z^4 = 0 \implies z = \exp\left(i\frac{\pi}{4}+k\frac{\pi}{2}\right), \quad k = 0,1,2,3$$

If we check which poles lie inside C, we see the only poles we have to consider are $z = 0, \pm 1$. We can now compute the residues and use the residue theorem. Note that all the poles are simple, thus we have

$$\text{Res}(f,0) = \lim_{z\to 0} z\frac{\cos(\pi z)}{\sin(\pi z)(1+z^4)} = \lim_{z\to 0}\frac{z}{\sin(\pi z)} = \frac{1}{\pi}$$

$$\text{Res}(f,\pm 1) = \lim_{z\to \pm 1}(z\pm 1)\frac{\cos(\pi z)}{\sin(\pi z)(1+z^4)} = \lim_{z\to \pm 1} -\frac{z\pm 1}{2\sin(\pi z)} = \frac{1}{2\pi}$$

Thus we have

$$\int_C \frac{\cos(\pi z)}{\sin(\pi z)(1+z^4)}dz = 2\pi i\,(\text{Res}(f,0)+\text{Res}(f,1)+\text{Res}(f,-1)) = 4i$$

□