Assignment 8

MATC34 – Complex Variables – Fall 2015

Solutions

Question 1 Find all singular points of the function

$$f(z) = \frac{z}{z^4 - 1} - \frac{\sin(2z)}{z^4}$$

If it is a pole, determine the order.

Solution We see the singular points are given by the roots of

$$z^4 - 1 = (z-1)(z+1)(z-i)(z+i) = 0 \quad \& \quad z^4 = 0$$

Clearly all the singular points from $z^4 - 1$ are $z = \pm 1, \pm i$ and they are simple poles. The pole at $z = 0$ is of third order since

$$\lim_{z\to 0} z^3 \frac{\sin(2z)}{z^4} = 2$$

$\square$

Question 2 Using Cauchy's Residue formula, find

$$\int_C \frac{\cot(\pi z)}{1+z^4} dz$$

where C is the positively oriented boundary of the rectangle with vertices at $(3 \pm i)/2$, $(-1 \pm i)/2$.

Solution We first will find the singular points of the integrand,

$$\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)} = \infty \iff \sin(\pi z) = 0 \implies z = n, \quad n \in \mathbb{Z} \quad \& \quad 1 + z^4 = 0 \implies z = \frac{1 \pm i}{\sqrt{2}}, \frac{-1 \pm i}{\sqrt{2}}$$

We see that $z = 0, 1$ are the only poles that lie inside C. We also see they're both simple, so the residue theorem tells us

$$\int_C \frac{\cot(\pi z)}{1+z^4} dz = 2\pi i(\mathrm{Res}(f,0) + \mathrm{Res}(f,1)) = 2\pi i \left(\lim_{z\to 0} z \frac{\cot(\pi z)}{1+z^4} + \lim_{z \to 1} (z-1) \frac{\cot(\pi z)}{1+z^4} \right) = 2\pi i \left(\frac{1}{\pi} + \frac{1}{2\pi} \right) = 3i$$

$\square$

Question 3 Using the reside at $z = \infty$, find the following integral

$$\int_{|z|=4} \frac{z^7}{(z-3)(z^5-1)} dz, \quad \int_{|z|=4} \frac{1}{(z-3)(z^5-1)} dz, \quad \int_{|z|=2} \frac{1}{(z-3)(z^5-1)} dz$$

Solution We substitute $w = 1/z$, $dw = -dz/z^2$ into the integrands to obtain

$$\int_{|w|=1/4} \frac{1}{w^3(1-3w)(1-w^5)} dw, \quad \int_{|w|=1/4} \frac{w^4}{(1-3w)(1-w^5)} dw, \quad \int_{|w|=1/2} \frac{w^4}{(1-3w)(1-w^5)} dw$$

where we preserved positive orientation. We see the only singular points inside the domains are $w = 0$ for the first integral, and $w = 1/3$ for the 3rd. We see that the pole at $w = 0$ is of third order, and the pole at $w = 1/3$ is simple in the 3rd integral. Thus we calculate the residues of these new integrands, call each integrand g_i, then

$$\text{Res}(g_1, 0) = \lim_{w \to 0} \frac{1}{2!} \frac{d^2}{dw^2} \frac{1}{(1-3w)(1-w^5)} = \lim_{w \to 0} \frac{1}{2!} \frac{d}{dw} \frac{3 + \mathcal{O}(w)}{(1-3w-w^5+3w^6)^2} = \lim_{w \to 0} \frac{9 + \mathcal{O}(\omega)}{(1-3w-w^5+3w^6)^3} = 9$$

$$\text{Res}(g_3, 1/3) = \lim_{w \to 1/3} (w - 1/3) \frac{w^4}{(1-3w)(1-w^5)} = -\frac{1}{(3^5-1)} = -\frac{1}{242}$$

The residue theorem now allows us to easily calculate the integrals since we've computed the residues, they are

$$\int_{|z|=4} \frac{z^7}{(z-3)(z^5-1)} dz = 18\pi i, \quad \int_{|z|=4} \frac{1}{(z-3)(z^5-1)} dz = 0, \quad \int_{|z|=2} \frac{1}{(z-3)(z^5-1)} dz = -\frac{2\pi i}{242} = -\frac{\pi i}{121}$$

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